# Residuated Lattices, Regular Languages, and Burnside Problem


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In this talk we are going to explore an interesting connection between the famous Burnside problem for groups, regular languages, and residuated lattices.

Let $\mathcal{K}$ be a finitely axiomatized class of residuated lattices. Recall that the usual way of proving decidability of universal theory for $\mathcal{K}$ is to establish that $\mathcal{K}$ has the finite embeddability property (FEP) [2, 3]. It turns out that the method of proving FEP for various classes of residuated structures used in [2, 3] is tightly connected to the generalized Myhill theorem [5] which characterizes regular languages as languages which are downward closed with respect to a dual well quasiorder compatible with concatenation of words.

Due to this connection, it is possible to convert some decidability proofs for classes of residuated lattices into sufficient regularity conditions for classes of languages. For example, one can use the proof of FEP for the variety of integral residuated lattices in order to show an already known fact that the class of languages closed under the following rule (corresponding to the well-known weakening rule)

$$uv \in L \implies uxv \in L\,,$$

contains only regular languages.

Let $m, n$ be natural numbers such that $m \neq n$. In this talk we will focus on the varieties $\mathcal{RL}^n_m$ of residuated lattices defined by $x^m \leq x^n$ for which we will discuss whether they possess the FEP and the above-mentioned connection with regular languages. For most of the cases the variety $\mathcal{RL}^n_m$ does not possess the FEP since its word problem is undecidable [6]. Disregarding further the trivial and simple cases, we end up with interesting situations, namely the cases when $m \geq 2$ and $n = 1$. This is the point where the Burnside problem for groups comes into play. Recall that the (bounded) Burnside problem for a given natural number $k$ asks whether the variety of groups defined by $x^k = 1$ is locally finite (see [1, 4]). It is known that the answer is affirmative for $k = 1, 2, 3, 4, 6$ and negative for odd $k \geq 665$ and any $k \geq 2^{13}$. For the remaining cases it is open.

Let $m \geq 2$. The main result of this talk shows that the method of proving the FEP for residuated lattices used in [2, 3] works for $\mathcal{RL}^1_m$ iff the Burnside problem for $m-1$ has an affirmative answer iff the class of languages closed under the following rule

$$ux_1v, ux_2v, \ldots, ux_mv \in L \implies ux_1x_2\ldots x_mv \in L\,,$$

contains only regular languages. Thus using the known results on the Burnside problem, we can infer that the varieties $\mathcal{RL}^1_m$ for $m = 2, 3, 4, 5, 7$ has the FEP.

Note that the main result gives us also an interesting connection between regular languages and the Burnside problem. Thus it is an appealing question if the theory of regular languages can help to solve the Burnside problem for the remaining open cases.


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