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Iterated contraction of propositions and conditionals under the principle of conditional preservation

Gabriele Kern-Isberner[1], Tanja Bock[2], Kai Sauerwald[3], and Christoph Beierle[4]

[1] TU Dortmund, Dortmund, Germany
gabriele.kern-isberner@cs.uni-dortmund.de
[2] TU Dortmund, Dortmund, Germany
tanja.bock@tu-dortmund.de
[3] University of Hagen, Hagen, Germany
kai.sauerwald@fernuni-hagen.de , ORCiD: 0000-0002-1551-7016
[4] University of Hagen, Hagen, Germany
christoph.beierle@fernuni-hagen.de

Abstract

Research on iterated belief change has focussed mostly on belief revision, only few papers have addressed iterated belief contraction. Most prominently, Darwiche and Pearl published seminal work on iterated belief revision the leading paradigm of which is the so-called principle of conditional preservation. In this paper, we use this principle in a thoroughly axiomatized form to develop iterated belief contraction operators for Spohn's ranking functions. We show that it allows for setting up constructive approaches to tackling the problem of how to contract a ranking function by a proposition or a conditional, respectively, and that semantic principles can also be derived from it for the purely qualitative case.

1 Introduction

AGM belief change [1] deals with the problem of how to change (propositional) beliefs of an agent in a rational way when the agent gets to know new information. The authors distinguished between three different forms of belief change: *Expansion* is the most simple form of belief change that should be used if the new information does not conflict with previously held beliefs; in this case, the new information should be simply added to the prior beliefs (under logical closure). In contrast, revision and contraction affect the prior beliefs. In case of *revision*, the new information conflicts with previously held beliefs, so some beliefs have to be given up to allow for the integration of the new informaion. *Contraction* means the process of eliminating some prior beliefs without adopting new beliefs. In the AGM framework, revision and contraction are dual operations because either one can be defined in terms of the other by the so-called Levi and Harper identities [9], so one might focus on any of them. As was shown [11], AGM revisions can be easily obtained by using total preorders on possible worlds in that revised beliefs are specified by the minimal (i.e., most plausible) models of the new information. However, while total preorders prove to be crucial ingredients to AGM revision, the revision process just yields propositional beliefs as output, hence does not provide a revised preorder for a successive revision step. In their seminal paper [4], Darwiche and Pearl expanded AGM belief revision [1] to the level of epistemic states represented by total preorders on possible worlds, thereby providing a base framework for iterated revision in the sense that the objects of the revision process are now total preorders so that the outcome of the process is also an epistemic state which might again be used for revision. They illustrated their approach by using Spohn's ranking functions [20, 22] that have become one of the standard frameworks for belief revision.

A bit surprizingly, the work on iterated revision has not initiated similar efforts to come up with basics for iterated contraction, i.e., contraction of total preorders by propositional beliefs. Except from few works in the last decade [21, 17, 10, 18],

it was only recently that Konieczny and Pino Pérez [16] proposed a close counterpart to Darwiche and Pearl's work [4]. Their main theorem

[16] characterizes iterated contraction in terms of conditions that the total preorders associated with prior and posterior epistemic states, respectively, should fulfill, very similar to what Darwiche and Pearl did for iterated revision. However, the principle of conditional preservation that principally inspired Darwiche and Pearl (beyond AGM's principle of minimal (propositional) change) was never mentioned in [16].

Indeed, while the connection between revision of epistemic states Ψ and conditionals $(B|A)$ representing a plausible rule "If A then usually B" is very obvious thanks to the Ramsey test [19] – Ψ accepts $(B|A)$ iff $\Psi * A \models B$, where $\Psi * A$ is the epistemic state resulting from revising Ψ by A –, the relevance of conditional beliefs for belief contraction is not so clear at first sight. But since in the framework of belief revision and nonmonotonic reasoning [8], conditionals are usually evaluated by considering minimal (i.e., most plausible) models of their antecedent, often according to total preorders, every change of a total preorder also affects conditional beliefs. Furthermore, same as for iterated revision, unjustified changes of conditional beliefs should also be avoided for iterated contraction. While one might consider the conditions in the representation theorem given by Konieczny and Pino Pérez [16, Theorem 4] as an implicit implementation of the basic ideas of [4] for iterated contraction, the role of conditional preservation for iterated contraction is still unexplored.

In this paper, we propose an approach to iterated contraction that makes use of the more comprehensive axiomatization of the principle of conditional preservation of [13, 14] (which implies the one of [4]) that can handle revision of so-called conditional valuation functions by sets of conditional beliefs and therefore provides guidelines for much more complex change processes than those of plain propositional beliefs. Conditional valuation functions are based on some quantitative structure for aggregating and comparing beliefs, such as probability functions, possibility distributions [6], or Spohn's ranking functions [20, 22]. We choose Spohn's ranking functions to elaborate our approach to iterated contraction from the principle of conditional preservation of [13, 14], and thanks to the generality of that belief change framework, we address iterated contraction of sets of conditionals right from the start. From this, we obtain results for iterated contraction of a single conditional, which can also be used to deal with iterated contraction by a single proposition. This is the context of [16], and here we show that our general principle of conditional preservation for iterated contraction, elaborated for ranking functions, provides a base for deriving all axioms that have been proposed for iterated contraction in [16]. More precisely, together with AGM-style postulates and a minor, intuitive postulate for iterated contraction, all axioms of [16] can be motivated by this fundamental principle.

After briefly fixing required formal preliminaries, we discuss related work on interated belief change (Sec. 3) and present the basic ideas of belief change under the principle of conditional preservation (Sec. 4). In Sec. 5 and 6, we address iterated contraction by a single proposition or conditional under conditional preservation and prove its properties, and in Sec. 7 we conclude and point out further work.

2 Formal preliminaries

Let $\mathcal{L}$ be a finitely generated propositional language, with atoms $a, b, c, \ldots$, and with formulas $A, B, C, \ldots$. For conciseness of notation, we will omit the logical *and*-connector, writing AB instead of $A \wedge B$, and overlining formulas will indicate negation, i.e. $\overline{A}$ means $\neg A$. Let Ω denote the set of possible worlds over $\mathcal{L}$; Ω will be taken here simply as the set of all propositional interpretations over $\mathcal{L}$. $\omega \models A$ means that the propositional formula $A \in \mathcal{L}$ holds in the possible world $\omega \in \Omega$. By slight abuse of notation, we will use ω both for the model and the corresponding conjunction of all positive or negated atoms. Given a set of possible worlds $\Omega' \subseteq \Omega$, $\mathcal{T}(\Omega') = \{A \in \mathcal{L} \mid \omega \models A \text{ for all } \omega \in \Omega'\}$ denotes the set of formulas which are true in all elements of Ω'. By introducing a new binary operator $|$, we obtain the set $(\mathcal{L} \mid \mathcal{L}) = \{(B|A) \mid A, B \in \mathcal{L}\}$ of conditionals over $\mathcal{L}$. $(B|A)$ formalizes "*if A then usually B*" and establishes a plausible connection between the *antecedent* A and the *consequent* B. Conditionals with tautological antecedents are taken as plausible statements about the world. Following De Finetti [5], a conditional $(B|A)$ can be *verified* (*falsified*) by a possible world ω iff $\omega \models AB$ ($\omega \models A\overline{B}$). If $\omega \not\models A$, then we say the conditional is *not applicable* to ω.

Ordinal conditional functions (OCFs), (also called *ranking functions*) $\kappa : \Omega \to \mathbb{N} \cup \{\infty\}$ with $\kappa^{-1}(0) \neq \emptyset$, were introduced (in a more general form) first by [20]. They express degrees of plausibility of propositional formulas A by specifying degrees of disbeliefs of their negations $\overline{A}$. More formally, we have $\kappa(A) := \min\{\kappa(\omega) \mid \omega \models A\}$, so that $\kappa(A \vee B) = \min\{\kappa(A), \kappa(B)\}$. Hence, due to $\kappa^{-1}(0) \neq \emptyset$, at least one of $\kappa(A), \kappa(\overline{A})$ must be 0. With $[\![\kappa]\!] = \{\omega \mid \kappa(\omega) = 0\}$, we denote the minimal models of κ. In this paper, to avoid technical peculiarities, we only consider OCFs $\kappa : \Omega \to \mathbb{N}$ within a finite range; an extension of our approach to the general case is straightforward. A conditional $(B|A)$ is accepted in the epistemic state represented by κ, written as $\kappa \models (B|A)$, iff $\kappa(AB) < \kappa(A\overline{B})$, i.e. iff AB is more plausible than $A\overline{B}$.

In more general settings, *epistemic states* Ψ will be represented by a total preorder $\preceq_\Psi$ on Ω which is most suitable in the context of belief revision (cf. Section 3). In a natural way $\preceq_\Psi$ can be lifted to a total preorder on the set of propositions via $A \preceq_\Psi B$ iff there is a $\omega_1 \in \mathit{Mod}(A)$ such that $\omega_1 \preceq_\Psi \omega_2$ for all $\omega_2 \in \mathit{Mod}(B)$. As usual, $\omega_1 \prec_\Psi \omega_2$ iff $\omega_1 \preceq_\Psi \omega_2$ and not $\omega_2 \preceq_\Psi \omega$, and $\omega_1 =_\Psi \omega_2$ iff both $\omega_1 \preceq_\Psi \omega_2$ and $\omega_2 \preceq_\Psi \omega_1$. Note that also OCFs κ induce total preorders on Ω via $\omega_1 \preceq_\kappa \omega_2$ iff $\kappa(\omega_1) \leqslant \kappa(\omega_2)$, so everything we state on general epistemic states will apply to OCFs, but OCFs allow for more expressive statements because of their usage of natural numbers and the corresponding arithmetics. If $\Omega' \subseteq \Omega$, then $\min(\preceq_\Psi, \Omega') = \{\omega_1 \in \Omega' \mid \omega_1 \preceq_\Psi \omega_2 \text{ for all } \omega_2 \in \Omega'\}$ denotes the set of minimal models in Ω' under Ψ. If $\Omega' = \Omega$, then we simply write $\min(\preceq_\Psi)$ instead of $\min(\preceq_\Psi, \Omega)$. If $A \in \mathcal{L}$, then $\min(\preceq_\Psi, A) = \min(\preceq_\Psi, \mathit{Mod}(A))$. The minimal models of an epistemic state form its associated belief set: $\mathit{Bel}(\Psi) = \mathcal{T}(\min(\Psi))$, i.e., the agent believes exactly the propositions that are valid in all most plausible models. For an OCF κ, we have accordingly $\mathit{Bel}(\kappa) = \mathcal{T}([\![\kappa]\!])$; thus, a proposition A is believed under κ if $\kappa(\overline{A}) > 0$ (which implies particularly $\kappa(A) = 0$).

3 Related work on iterated belief change

AGM theory [1] deals with belief revision in the context of belief sets, i.e., deductively closed sets of propositions, which are to represent a basic epistemic state. Whatever the type of epistemic state is, the question which epistemic structure is needed to guarantee that belief revision complies with AGM theory has been answered by [11] and [4]: AGM revision of an epistemic state Ψ can be ensured by assuming that a so-called faithful ranking underlies Ψ such that the revised beliefs can be computed from minimal models according to the ranking. Here, a faithful ranking is a total preorder $\preceq_\Psi$ on the possible worlds that is assigned to Ψ in such a way that the minimal models of $\preceq_\Psi$, denoted by $\min(\preceq_\Psi)$, are precisely the models of the belief set $K = Bel(\Psi)$ associated with Ψ.

Proposition 1 ([4]). *A revision operator $\star$ that assigns a posterior epistemic state $\Psi \star A$ to a prior state Ψ and a proposition A is an AGM revision operator for epistemic states iff there exists a faithful preorder $\preceq_\Psi$ for an epistemic state Ψ with associated belief set $K = Bel(\Psi)$, such that for every proposition C it holds that:*

$$K \star C = Bel(\Psi \star C) = \mathcal{T}(\min(\preceq_\Psi, C))$$

This proposition allows us to study AGM-style revisions by focussing on total preorders. As pointed out by Darwiche, Pearl and others some of the revisions characterised by Proposition 1 lead to unintuitive results in the case of iterated revision. An iterative revision should fulfill further postulates, especially those that ensure that the ordering of specific worlds is kept:

Proposition 2 ([4]). *Let $\star$ be an AGM revision operator for epistemic states Ψ with corresponding faithful preorder $\preceq_\Psi$. Then $\star$ is an iterative revision operator in the sense of [4] iff for every proposition C it holds that:*

$$\text{if } \omega_1, \omega_2 \models C, \text{ then } \omega_1 \preceq_\Psi \omega_2 \text{ iff } \omega_1 \preceq_{\Psi\star C} \omega_2 \tag{CR1}$$

$$\text{if } \omega_1, \omega_2 \models \overline{C}, \text{ then } \omega_1 \preceq_\Psi \omega_2 \text{ iff } \omega_1 \preceq_{\Psi\star C} \omega_2 \tag{CR2}$$

$$\text{if } \omega_1 \models C \text{ and } \omega_2 \models \overline{C}, \text{ then } \omega_1 \prec_\Psi \omega_2 \text{ implies } \omega_1 \prec_{\Psi\star C} \omega_2 \tag{CR3}$$

$$\text{if } \omega_1 \models C \text{ and } \omega_2 \models \overline{C}, \text{ then } \omega_1 \preceq_\Psi \omega_2 \text{ implies } \omega_1 \preceq_{\Psi\star C} \omega_2 \tag{CR4}$$

This approach by Darwiche and Pearl has been widely accepted, and is the basis for many results on iterated belief revision. In comparison to iterated belief revision, the case of iterated belief contraction is less examined. Spohn [21], Hansson [10], Nayak et al. [17, 18], and [2] consider special aspects of iterated belief contraction, but to the best of our knowledge the recent paper by Konieczny and Pino Pérez [16] is the only work on propositional iterated contraction that addresses the general case. Similar to the work of Darwiche and Pearl, Konieczny and Pino Pérez adapt a set of AGM contraction postulates for the propositional case; these propositional contraction postulates where proposed by Caridroit, Konieczny and Marquis [3]. We recall the postulates from Konieczny and Pino Pérez [16], where $-$ is a contraction operator that assigns a posterior epistemic state $\Psi - C$ to a prior state Ψ and a proposition C:

$$Bel(\Psi) \models Bel(\Psi - C) \tag{AGMes-1}$$

$$\text{If } Bel(\Psi) \not\models C, \text{ then } Bel(\Psi - C) \models Bel(\Psi) \tag{AGMes-2}$$

$$\text{If } Bel(\Psi - C) \models C, \text{ then } C \equiv \top \tag{AGMes-3}$$

$$Bel(\Psi - C) \wedge C \models Bel(\Psi) \tag{AGMes-4}$$

$$\text{If } C \equiv B, \text{ then } Bel(\Psi - C) \equiv Bel(\Psi - B) \tag{AGMes-5}$$

$$Bel(\Psi - (C \wedge B)) \models Bel(\Psi - C) \vee Bel(\Psi - B) \tag{AGMes-6}$$

$$\text{If } Bel(\Psi - (C \wedge B)) \not\models C, \text{ then } Bel(\Psi - C) \models Bel(\Psi - (C \wedge B)) \tag{AGMes-7}$$

For an explanation of these postulates we refer to the article of Caridroit et al. [3], where the authors explain the corresponding postulates for AGM contraction on belief sets for the propositional case. Konieczny and Pino Pérez give a characterisation theorem for the postulates (AGMes-1) to (AGMes-7) in terms of faithful rankings [16].

Proposition 3 ([16])**.** *A contraction operator* $-$ *that assigns a posterior epistemic state* $\Psi - A$ *to a prior state* Ψ *and a proposition* A *fulfills* (AGMes-1) *to* (AGMes-7) *iff there exists a faithful preorder* $\preceq_\Psi$ *for an epistemic state* Ψ *such that for every proposition* C *it holds that* $Bel(\Psi - A) = \mathcal{T}(\min(\preceq_\Psi) \cup \min(\preceq_\Psi, \neg C))$*, i.e.*

$$[\![\Psi - A]\!] = [\![\Psi]\!] \cup \min([\![\neg C]\!], \preceq_\Psi) \tag{1}$$

Proposition 3 makes clear that the (AGMes-) postulates for contraction constrain which worlds are in the lowest layer, but they do not constrain the ordering of worlds in higher layers. Similar to the case of iterated revision this is not enough to ensure intuitive iterative belief contractions. This problem has been addressed in [16] by introducing a set of postulates the semantic versions of which are shown in the following proposition:

Proposition 4 ([16])**.** *Let* $-$ *be an AGM contraction operator for epistemic states* Ψ *with corresponding faithful preorder* $\preceq_\Psi$*. Then* $-$ *is an iterative contraction operator in the sense of [16] iff for every proposition* C *it holds that:*

$$\text{if } \omega_1, \omega_2 \models C, \text{ then } \omega_1 \preceq_\Psi \omega_2 \text{ iff } \omega_1 \preceq_{\Psi - C} \omega_2 \tag{IC1}$$

$$\text{if } \omega_1, \omega_2 \models \overline{C}, \text{ then } \omega_1 \preceq_\Psi \omega_2 \text{ iff } \omega_1 \preceq_{\Psi - C} \omega_2 \tag{IC2}$$

$$\text{if } \omega_1 \models \overline{C} \text{ and } \omega_2 \models C, \text{ then } \omega_1 \prec_\Psi \omega_2 \text{ implies } \omega_1 \prec_{\Psi - C} \omega_2 \tag{IC3}$$

$$\text{if } \omega_1 \models \overline{C} \text{ and } \omega_2 \models C, \text{ then } \omega_1 \preceq_\Psi \omega_2 \text{ implies } \omega_1 \preceq_{\Psi - C} \omega_2 \tag{IC4}$$

Among these postulates for *I̲terated C̲ontraction*, the postulates (IC1) and (IC2), called *rigidity conditions* in [16] and *ordered preservation conditions* in [18], ensure that the order of worlds does not change if they are equivalent in the perspective of contraction. The postulates (IC3) and (IC4), called *non-worsening conditions* in [16], enforce that no world that contradicts the contracted information is getting relatively more plausible after the contraction than a world that was already more plausible before and does not contradict the contracted information. For instance, if we contract with a and $\overline{a}b \preceq_\kappa ab$ holds, then this order has to be kept, i.e. $\overline{a}b \preceq_{\kappa - a} ab$. Note that (IC1) and (IC2) directly correspond to (CR1) and (CR2), while (IC3) and (IC4) can be seen as dual variants of (CR3) and (CR4) obtained essentially by exchanging C and $\overline{C}$, reflecting the inverse aim of contraction with respect to revision.

Another approach to epistemic change is developed in [13] where belief change is considered in a very general and advanced form: Epistemic states are revised by sets of *conditionals*, thus exceeding the classical AGM theory and related approaches which only deal with pieces, or sets of propositional beliefs. In the following, we will elaborate how this aproach can be employed for iterated contraction of propositions and conditionals, thereby also covering iterated contraction in the sense of [16].

4 Iterated belief change under the principle of conditional preservation

The principle of conditional preservation proposed in [14] for iterated conditional revision is an invariance condition that aims to maintain conditional relationships in the prior epistemic state if there is no reason to alter them, and any reason for changes must be explained by obvious differences with respect to the new information. So far, this principle has only been applied to revision scenarios, but in fact, preservation of conditional beliefs is a principle for general epistemic change processes which is as basic as the paradigm of minimal (propositional) change in the AGM framework. In the following, we formalize the principle of conditional preservation for general multiple change operators $\circ$ in the context of Spohn's ranking functions, carry over the characterization theorem from [14], and apply this to iterated contractions by single pieces of conditional information.

The theory of conditional structures [13, 14] provides a suitable basis to axiomatize the principle of conditional preservation. Conditional structures are kind of footprints that sets of conditionals leave on possible worlds. They base on the three-valued approach to conditionals by De Finetti [5], representing verification, falsification, and non-applicability by abstract symbols that are assigned to conditionals (or propositions) in a general information set. Since different conditionals are assigned different symbols, each conditional is treated as an *independent piece of information*. More formally, let $\mathcal{R} = \{(B_1|A_1), \ldots, (B_n|A_n)\} \subseteq (\mathcal{L} \mid \mathcal{L})$ be a finite set of conditionals, and let $\mathbf{a}_1^+, \mathbf{a}_1^-, \ldots, \mathbf{a}_n^+, \mathbf{a}_n^-$ be distinct algebraic symbols that are used as generators of a (free abelian[1]) group [7] $\mathcal{F}_\mathcal{R} = \langle \mathbf{a}_1^+, \mathbf{a}_1^-, \ldots, \mathbf{a}_n^+, \mathbf{a}_n^- \rangle$. In short, this group structure provides us with a multiplication (written as juxtaposition) that we can relate to the summation of κ-values, and with a neutral element 1 we can use for symbolizing non-applicability. Furthermore, free abelian ensures a parallel handling of the conditionals (without any order of application assumed) as well as independence between different conditionals (by forbidding cancellations between different symbols). For each $i, 1 \leqslant i \leqslant n$, we define a function $\sigma_i = \sigma_{(B_i|A_i)} : \Omega \to \mathcal{F}_\mathcal{R}$ by setting

$$\sigma_i(\omega) := \begin{cases} \mathbf{a}_i^+ & \text{if} \quad \omega \models A_i B_i \quad \text{(verification)} \\ \mathbf{a}_i^- & \text{if} \quad \omega \models A_i \overline{B_i} \quad \text{(falsification)} \\ 1 & \text{if} \quad \omega \models \overline{A_i} \quad \text{(non-applicability)} \end{cases}$$

$\sigma_i(\omega)$ represents the manner in which the conditional $(B_i|A_i)$ applies to the possible world ω. The function $\sigma_\mathcal{R} : \Omega \to \mathcal{F}_\mathcal{R}$ given by

$$\sigma_\mathcal{R}(\omega) := \prod_{1 \leqslant i \leqslant n} \sigma_i(\omega) = \prod_{\substack{1 \leqslant i \leqslant n \\ \omega \models A_i B_i}} \mathbf{a}_i^+ \prod_{\substack{1 \leqslant i \leqslant n \\ \omega \models A_i \overline{B}_i}} \mathbf{a}_i^-$$

describes the all-over effect of $\mathcal{R}$ on ω. Furthermore, $\sigma_\mathcal{R}(\omega)$ is called the *conditional structure of ω with respect to $\mathcal{R}$*. Since $\mathcal{F}_\mathcal{R}$ is a free (abelian) group, the conditional structures of worlds are uniquely determined by their σ_i-components and hence by their logical relation to each conditional: For any two worlds ω_1, ω_2, we have

$$\sigma_\mathcal{R}(\omega_1) = \sigma_\mathcal{R}(\omega_2) \quad \text{iff} \quad \sigma_i(\omega_1) = \sigma_i(\omega_2) \quad \text{for all } i,\ 1 \leqslant i \leqslant n. \tag{2}$$

Conditional structures also work for propositions $A_i \equiv (A_i|\top) \in \mathcal{R}$. Propositions can only be verified resp. satisfied, or falsified. There is no non-applicability in this case, so $\sigma_i(\omega) \in \{\mathbf{a}_i^-, \mathbf{a}_i^+\}$ for each ω. Nevertheless, conditional structures are helpful to distinguish between the effects of different pieces of information also in this case.

Example 5 As a running example we use the knowledge base $\mathcal{R}_{pen} = \{r_1, r_2, r_3\}$ which consists of conditionals describing our knowledge about penguins. Let $\Sigma = \{p, b, f\}$ be the set of atoms with p meaning "something is a penguin", b "something is a bird" and f "something is able to fly". Then we can model that "birds normally fly" with the conditional $r_1 = (f|b)$, "penguins normally do not fly" with $r_2 = (\overline{f}|p)$ and "penguins are normally birds" with $r_3 = (b|p)$. Assigning symbols $\mathbf{a}_i^+, \mathbf{a}_i^-$ to any r_i, we compute the following conditional structures for some

[1]Free abelian groups have no relations except for those induced by commutativity.

of the possible worlds:

$$\sigma_{\mathcal{R}}(pbf) = \mathbf{a}_1^+\mathbf{a}_2^-\mathbf{a}_3^+, \qquad \sigma_{\mathcal{R}}(p\overline{b}\,\overline{f}) = \mathbf{a}_2^+\mathbf{a}_3^-, \qquad \sigma_{\mathcal{R}}(\overline{p}\,\overline{b}f) = 1. \qquad \blacksquare$$

With these notations, the principle of conditional preservation of [14] can be stated concisely for multiple iterated change operators $\circ$ that assign a posterior ranking function $\kappa \circ \mathcal{R}$ to a prior ranking function κ and a set $\mathcal{R}$ of conditionals, representing the new information:

($\mathbf{PCP}_{\circ}^{ocf}$) Let $\circ$ be a change operation on OCFs, and let $\mathcal{R} = \{(B_1|A_1), \ldots, (B_n|A_n)\} \subseteq (\mathcal{L} \mid \mathcal{L})$ be a finite set of conditionals. If two multisets of possible worlds $\Omega = \{\omega_1, \ldots, \omega_m\}$ and $\Omega' = \{\omega_1', \ldots, \omega_m'\}$ with the same cardinality fulfill $\prod_{j=1}^{m} \sigma_{\mathcal{R}}(\omega_j) = \prod_{j=1}^{m} \sigma_{\mathcal{R}}(\omega_j')$, i.e., for each conditional $(B_i|A_i)$ in $\mathcal{R}$, Ω and Ω' show the same number of verifications resp. falsifications, then prior κ and posterior $\kappa^{\circ} = \kappa \circ \mathcal{R}$ are balanced by:

$$\begin{aligned}&(\kappa(\omega_1) + \ldots + \kappa(\omega_m)) - (\kappa(\omega_1') + \ldots + \kappa(\omega_m'))\\ &\quad = (\kappa^{\circ}(\omega_1) + \ldots + \kappa^{\circ}(\omega_m)) - (\kappa^{\circ}(\omega_1') + \ldots + \kappa^{\circ}(\omega_m'))\end{aligned}$$

Of course, ($\mathrm{PCP}_{\circ}^{ocf}$) requires the ranking framework to provide basic arithmetic features. Nevertheless, in the following, we will consider simple applications of ($\mathrm{PCP}_{\circ}^{ocf}$) to obtain purely qualitative derivatives of it. First, if $|\Omega| = |\Omega'| = 1$, we have the following simple, straightforward consequence:

($\mathbf{PCP1}_{\circ}^{ocf}$) If two possible worlds $\omega_1, \omega_2 \in \Omega$ fulfill $\sigma_{\mathcal{R}}(\omega_1) = \sigma_{\mathcal{R}}(\omega_2)$, i.e., ω_1, ω_2 verify resp. falsify exactly the same conditionals in $\mathcal{R}$, then $\kappa^{\circ}(\omega_1) - \kappa(\omega_1) = \kappa^{\circ}(\omega_2) - \kappa(\omega_2)$.

($\mathrm{PCP1}_{\circ}^{ocf}$) claims that the amount of change between prior epistemic state κ and the posterior epistemic state $\kappa \circ \mathcal{R} = \kappa^{\circ}$ depends only on the conditionals in the new information set $\mathcal{R}$, more precisely, on the conditional structure of the respective world. ($\mathrm{PCP1}_{\circ}^{ocf}$) can also be written in the form

$$\kappa \circ \mathcal{R}(\omega_1) - \kappa \circ \mathcal{R}(\omega_2) = \kappa(\omega_1) - \kappa(\omega_2) \tag{3}$$

and then claims that the epistemic distance between two worlds that behave the same with respect to the new information $\mathcal{R}$ should be kept constant under change. On the qualitative level, the postulate ($\mathrm{PCP1}_{\circ}^{ocf}$) implies that $\omega_1 \preceq_{\kappa \circ \mathcal{R}} \omega_2$ iff $\omega_1 \preceq_{\kappa} \omega_2$. Replacing $\preceq_{\kappa}$ by a more general faithful ranking of an epistemic state $\preceq_{\Psi}$ yields a qualitative (semantic) principle for epistemic states Ψ which are specified by a total preorder $\preceq_{\Psi}$ on Ω:

($\mathbf{QPCP1}_{\circ}^{sem}$) Let $\mathcal{R} = \{(B_1|A_1), \ldots, (B_n|A_n)\} \subseteq (\mathcal{L} \mid \mathcal{L})$ be a finite set of conditionals. If $\omega_1, \omega_2 \in \Omega$, then $\sigma_{\mathcal{R}}(\omega_1) = \sigma_{\mathcal{R}}(\omega_2)$ implies $\omega_1 \preceq_{\Psi} \omega_2$ iff $\omega_1 \preceq_{\Psi \circ \mathcal{R}} \omega_2$.

Now, we transfer the characterisation of the change of ranking functions under the principle of conditional preservation (i.e., in the context of revision by sets of conditionals) from [13] to the case of multiple conditional change:

Theorem 6. *Let $\mathcal{R} = \{(B_1|A_1), \ldots, (B_n|A_n)\} \subseteq (\mathcal{L} \mid \mathcal{L})$ be a finite set of conditionals, and let $\kappa \circ \mathcal{R} = \kappa^{\circ}$ be a belief change of κ by $\mathcal{R}$. Then this change satisfies ($PCP_{\circ}^{ocf}$) iff there are rational[2] numbers $\kappa_0, \gamma_i^+, \gamma_i^-$, $1 \leqslant i \leqslant n$ such that*

$$\kappa^{\circ}(\omega) = \kappa_0 + \kappa(\omega) + \sum_{\substack{1 \leqslant i \leqslant n \\ \omega \models A_i B_i}} \gamma_i^+ + \sum_{\substack{1 \leqslant i \leqslant n \\ \omega \models A_i \overline{B_i}}} \gamma_i^- . \tag{4}$$

Iterated belief change operators of the form (4) *are called* c-change operators*.*

[2] Note that indeed, $\kappa_0, \gamma_i^+, \gamma_i^-$ can be rational, but κ° has to satisfy the requirements for OCF, in particular, all $\kappa^{\circ}(\omega)$ must be non-negative integers.

The proof of this theorem is exactly the same as the proof of Theorem 4.6.1 in [13] since no specific information on the kind of change was used there. γ_i^+ and γ_i^- are constants associated with each conditional affecting verifying and falsifying worlds in a uniform way, and κ_0 is a normalizing constant ensuring that κ° is an OCF, i.e. there is at least one world ω such that $\kappa^\circ(\omega) = 0$.

Note that the way the principle of conditional preservation was axiomatized for iterated revision by sets of conditionals in [13] consists of two independent steps: First, ($\mathrm{PCP}_\circ^{ocf}$) was established for revisions, and afterwards combined with the success condition for revision. Hence, to obtain a schema for an effective iterated change operator from (4), one has to further impose a suitable success condition that specifies what the change operator is meant to achieve.

While the general case of this kind of belief change was already examined [14], let us now focus on revising an OCF κ by a singleton set $\mathcal{R} = \{(B|A)\}$ as a special case of this framework. Hence the success postulate in this case is $\kappa^* \models (B|A)$ with $\kappa^* = \kappa * (B|A)$, which is equivalent to

$$\kappa^*(AB) < \kappa^*(A\overline{B}). \tag{5}$$

In this case combining the principle of conditional preservation represented by equation (4) with the success postulate given by (5) yields the constraint (see e.g. [15] for details):

$$\gamma^- - \gamma^+ > \kappa(AB) - \kappa(A\overline{B}) \tag{6}$$

Given this, we can simplify κ_0 from equation (4) to $\kappa_0 = -\min\{\gamma^+ + \kappa(AB), \kappa(\overline{A})\}$. Thus, when specializing Theorem 6 to the case of revision by a single conditional $(B|A)$, equation (4) simplifies to

$$\kappa * (B|A)\,(\omega) = -\min\{\gamma^+ + \kappa(AB), \kappa(\overline{A})\} + \kappa(\omega) + \begin{cases} \gamma^+ & \textit{if}\ \omega \models AB \\ \gamma^- & \textit{if}\ \omega \models A\overline{B} \\ 0 & \text{otherwise} \end{cases} \tag{7}$$

Obviously, the postulates for conditional revision as elaborated by Kern-Isberner [14] are also fulfilled by revisions in this special case; a detailed elaboration of the properties of revisions under the principle of conditional preservation by a single conditional can be found in [12, 15].

5 Iterated contraction by a single proposition resp. conditional

We will now show how we can derive a general approach for iterated contraction operations from Theorem 6 (Sec. 5.1), refine the general setting to two special cases (Sec. 5.2), and in Sec. 5.3 we will focus on iterated contraction by a single proposition.

5.1 General case

In the case that we focus on in this paper, $\circ = \ominus$ is an iterated contraction operator handling a single piece of conditional information, i.e., $\mathcal{R} = \{(B|A)\}$, and returning a c-contracted ranking function $\kappa^\ominus = \kappa\ominus(B|A)$. Note that this also covers the case of contracting a single proposition A when A is identified with $(A|\top)$. In this setting, *success* reads $\kappa^\ominus \not\models (B|A)$, which is equivalent to $\kappa^\ominus(A\overline{B}) \leqslant \kappa^\ominus(AB)$. For this case, equation (4) from Theorem 6 reads

$$\kappa^\ominus(\omega) = \kappa_0 + \kappa(\omega) + \begin{cases} \gamma^+ & \textit{if}\ \ \omega \models AB \\ \gamma^- & \textit{if}\ \ \omega \models A\overline{B} \\ 0 & \textit{if}\ \ \omega \models \overline{A} \end{cases}, \tag{8}$$

where the constraint $\kappa^{\ominus}(\top) = 0$ together with the success condition immediately yields

$$\kappa_0 = -\min\{\gamma^- + \kappa(A\overline{B}), \kappa(\overline{A})\}. \tag{9}$$

By applying the success condition to (8), we obtain the following crucial condition for the parameters γ^+, γ^- to provide successful contraction operators:

$$\gamma^- - \gamma^+ \leqslant \kappa(AB) - \kappa(A\overline{B}). \tag{10}$$

Note the similarities, but also differences between (8), (9), and (10), on the one hand, and (6), (7) on the other. Summarizing so far, the equations (8), (9), and (10) provide a compact schema for iterated c-contractions of ranking functions by a conditional $(B|A)$. In dependance of the parameters γ^+, γ^-, we obtain a whole family of c-contractions, all respecting the principle of conditional preservation ($\mathrm{PCP}_{\ominus}^{ocf}$) for iterated contraction.

5.2 Special cases of c-contractions

In the following, we will elaborate on specific iterated c-contraction operators based on the schema for singleton c-contractions (equation (8) - (10)) which we call type α and type β c-contractions:

Type α c-contraction: The $A\overline{B}$ worlds are made more plausible while the AB worlds are not affected, i.e., $\gamma^- \leqslant 0, \gamma^+ = 0$. In the following we use $\kappa^{\ominus}_{\alpha,\gamma^-}$ to indicate a c-contracted ranking function of type α with a concrete impact value γ^-.

Type β c-contraction: The AB worlds are made less plausible while the $A\overline{B}$ worlds are not affected, i.e., $\gamma^+ \geqslant 0, \gamma^- = 0$. We use $\kappa^{\ominus}_{\beta,\gamma^+}$ for a c-contracted ranking function of type β with a specific γ^+.

Note that equation (10) immediately yields $\gamma^- \leqslant \min\{0, \kappa(AB) - \kappa(A\overline{B})\}$ for type α c-contractions and $\gamma^+ \geqslant \max\{0, \kappa(A\overline{B}) - \kappa(AB)\}$ for type β c-contractions.

A desirable goal for any belief change operator is the minimal change paradigm, meaning that the beliefs only change as much as necessary. One way of achieving this is by doing nothing if the success condition is already satisfied, i.e. $\kappa^{\ominus} = \kappa$ if $\kappa \not\models (B|A)$. Another possibility is to change the beliefs that contradict the new information in a minimal way just so that the success condition is satisfied. If we look at contractions with minimal change, meaning that $|\gamma^+|$ and $|\gamma^-|$ are minimal, the constraints for both previously c-contraction types α and β can be further specified. We will now consider this second approach.

We consider at first type α c-contractions, where we have $\gamma^+ = 0$. The minimal value for γ^- is then $\gamma^-_{\mathbf{min}} = \min\{0, \kappa(AB) - \kappa(A\overline{B})\}$. If the conditional is accepted by the ranking function, i.e. $\kappa \models (B|A)$, we have $\kappa(AB) - \kappa(A\overline{B}) < 0$ and hence $\gamma^- = \kappa(AB) - \kappa(A\overline{B})$. This leads to the general form of minimal type α c-contractions:

$$\kappa^{\ominus}_{\alpha,\mathbf{min}}(\omega) = -\min\{\kappa(AB), \kappa(\overline{A})\} + \kappa(\omega) + \begin{cases} \kappa(AB) - \kappa(A\overline{B}) & \textit{if } \omega \models A\overline{B} \\ 0 & \text{otherwise} \end{cases} \tag{11}$$

In the case of $\kappa \not\models (B|A)$, we get $\gamma^- = 0$. So in the case of minimal type α c-contractions the paradigm of minimal change is fulfilled and we have $\kappa^{\ominus}_{\alpha,\mathbf{min}} = \kappa$.

Secondly we consider type β c-contractions, where we have $\gamma^- = 0$ which leads immediately to $\kappa_0 = -\min\{\kappa(A\overline{B}), \kappa(\overline{A})\}$. By choosing the minimal possible change for a type

ω	$\kappa(\omega)$	$\kappa^{\ominus}_{\alpha,-1}(\omega)$	$\kappa^{\ominus}_{\alpha,-3}(\omega)$	$\kappa^{\ominus}_{\beta,1}(\omega)$	$\kappa^{\ominus}_{\beta,3}(\omega)$
$b\,f\,p$	2	1	1	2	2
$b\,f\,\overline{p}$	0	0	2	0	0
$b\,\overline{f}\,p$	1	1	3	2	4
$b\,\overline{f}\,\overline{p}$	1	1	3	1	1
$\overline{b}\,f\,p$	4	3	3	4	4
$\overline{b}\,f\,\overline{p}$	0	0	2	0	0
$\overline{b}\,\overline{f}\,p$	2	2	4	3	5
$\overline{b}\,\overline{f}\,\overline{p}$	0	0	2	0	0

Table 1: Ranking functions for the simple penguin example. The ranking functions are computed by contracting with $(\overline{f}|p)$, i.e. $\kappa^{\ominus}_{\alpha|\beta,\gamma^+|\gamma^-} = \kappa \ominus (\overline{f}|p)$.

β c-contraction we get the constraint $\gamma^+_{\mathbf{min}} = \max\{0, \kappa(A\overline{B}) - \kappa(AB)\}$, which leads to the following general form of minimal type β c-contractions by adapting equation (8):

$$\kappa^{\ominus}_{\beta,\mathbf{min}}(\omega) = -\min\{\kappa(A\overline{B}), \kappa(\overline{A})\} + \kappa(\omega) + \begin{cases} \kappa(A\overline{B}) - \kappa(AB) & \textit{if } \omega \models AB \\ 0 & \text{otherwise} \end{cases} \tag{12}$$

In the case that the conditional is not accepted by the ranking function, we have $\gamma^+ = 0$ which leads to $\kappa_0 = 0$. With this we get the same equation as for a type α contraction, i.e. we have $\kappa^{\ominus}_{\beta,\mathbf{min}} = \kappa$.

Example 7 Consider the knowledge base $\mathcal{R}_{pen}$of Example 5 describing the connection about penguins and birds. We use the ranking function κ given in Table 1 which gives us $\kappa(p\overline{f}) = 1$ and $\kappa(pf) = 2$, so we have $\kappa \models (\overline{f}|p)$. Now we want to contract that penguins cannot fly.

If we apply our schema of c-contractions (see equation (8)) to this example, we obtain the following equation:

$$\kappa^{\ominus}_{\alpha,\gamma^-}(\omega) = -\min\{1, \gamma^- + 1, 0\} + \kappa(\omega) + \begin{cases} \gamma^- & \textit{if } \omega \models pf \\ 0 & \text{otherwise} \end{cases},$$

with $\gamma^- \leqslant -1$. By choosing $\gamma^- = -1$ we obtain a minimal type α c-contractions. This gives us the ranking function shown in the third column of Table 1. Furthermore we have the ranking function for $\gamma^- = -3$ with $\kappa_0 = 2$ in the fourth column.

A type β c-contraction makes the worlds ω with $\omega \models p\overline{f}$ less plausible, leading to

$$\kappa^{\ominus}_{\beta,\gamma^+}(\omega) = -\min\{\gamma^+ + 1, 1, 0\} + \kappa(\omega) + \begin{cases} \gamma^+ & \textit{if } \omega \models p\overline{f} \\ 0 & \text{otherwise} \end{cases},$$

with the restriction $\gamma^+ \geqslant 1$. Independent of the choice of γ^+ it is always $\kappa_0 = 0$ for this example. A minimal change is achieved by choosing $\gamma^+ = 1$, see fifth column in Table 1. Additionally we have also the ranking function for $\gamma^+ = 3$ in the sixth column of the table. ■

5.3 Iterated contraction by a single proposition

Equation (8) - (10) can be specialized for belief contraction by single propositions by identifying proposition A with the conditional $(A|\top)$. In contrast to contractions with conditionals, there are only two options, either a proposition A is true in a world ω or not. For a successful c-contraction with a proposition A, namely $\kappa \ominus A = \kappa^{\ominus}$, we must have $\kappa^{\ominus} \not\models A$. This is the case if $\kappa^{\ominus}(\overline{A}) \leqslant \kappa^{\ominus}(A)$ holds. Since $\kappa(A) = 0$ or $\kappa(\overline{A}) = 0$ holds for every ranking function we can conclude that $\kappa^{\ominus}(\overline{A}) = 0$. Transferring (9) to the propositional case, we get $\kappa_0 = -\min\{\gamma^+ + \kappa(A), \gamma^- + \kappa(\overline{A})\} = -\gamma^- - \kappa(\overline{A})$. By this, we can obtain a general schema for contracting by a proposition A:

$$\kappa^{\ominus}(\omega) = -\gamma^- - \kappa(\overline{A}) + \kappa(\omega) + \begin{cases} \gamma^+ & \text{if } \omega \models A \\ \gamma^- & \text{if } \omega \models \overline{A} \end{cases} = \kappa(\omega) - \kappa(\overline{A}) + \begin{cases} \gamma^+ - \gamma^- & \text{if } \omega \models A \\ 0 & \text{if } \omega \models \overline{A} \end{cases}, \quad (13)$$

such that $\gamma^- - \gamma^+ \leqslant \kappa(A) - \kappa(\overline{A})$. We call c-contractions of the form (13) *propositional c-contractions*. We will now check whether this specialized case fulfills the (AGMes-1) - (AGMes-7) postulates.

Theorem 8. *A propositional c-contraction as defined in* (13) *fulfills the* (AGMes-1) *-* (AGMes-7) *postulates iff for $\kappa(A) > 0$, it has the form*

$$\kappa^{\ominus}(\omega) = \kappa(\omega) + \begin{cases} \gamma^+ - \gamma^- & \text{if } \omega \models A \\ 0 & \text{if } \omega \models \overline{A} \end{cases} \quad \text{with} \quad \gamma^+ - \gamma^- > \kappa(\overline{A}) - \kappa(A) \quad (14)$$

while for $\kappa(A) = 0$, it has the form

$$\kappa^{\ominus}(\omega) = \kappa(\omega) + \begin{cases} 0 & \text{if } \omega \models A \\ -\kappa(\overline{A}) & \text{if } \omega \models \overline{A} \end{cases} \quad (15)$$

Proof. For the proof of Theorem 8 we use Proposition 3. A contraction operation satisfies (AGMes-1) - (AGMes-7) iff (1) holds, so we have to show that the models of the contracted ranking function $[\![\kappa - A]\!]$ consists of the previous models $[\![\kappa]\!]$ and the minimal models of $[\![\neg A]\!]$ regarding the total pre-order $\preceq_\kappa$. We will show this in two steps: First we prove that (14) and (15) fulfill (1) and then show that every c-contraction defined by equation (13) that fulfills (1) is of the form (14) or (15).

If we look at (14), where we have $\kappa(A) > 0$, it is possible that the worlds that satisfy A are changed whereas the rank of the other worlds remain the same. To ensure that it is a successful c-contraction it is necessary that $\kappa^{\ominus}(\overline{A}) = 0$ holds, which is only possible if $\kappa(\overline{A}) = 0$ holds. According to (14) we have the constraint $\gamma^+ - \gamma^- > -\kappa(A)$. If we look at the minimal models of the contracted ranking function we see that $[\![\kappa - A]\!] = \min([\![\overline{A}]\!], \preceq_\kappa) = [\![\kappa]\!]$. So for the first special form (1) is fulfilled.

Equation (15) describes a c-contraction that moves the worlds of $\overline{A}$ to the bottom of the ranking function and do not change the other worlds. So we have $\kappa^{\ominus}(\overline{A}) = 0$ after the contraction, meaning $\min([\![\overline{A}]\!], \preceq_\kappa) \subseteq [\![\kappa - A]\!]$. For worlds that satisfy A, we have $\kappa^{\ominus}(\omega) = 0$ only if the world had the rank in the original ranking function, i.e. $\kappa(\omega) = 0$. So in the case that $\kappa(A) = 0$ holds, we have $[\![\kappa - A]\!] = [\![\kappa]\!] \cup \min([\![\overline{A}]\!], \preceq_\kappa)$, so (1) is fulfilled.

We now show that every c-contraction, that is defined by (13) and fulfills Proposition 3, is of the form (14) or (15). Let $\kappa \ominus A = \kappa^{\ominus}$ be such a c-contraction that fulfills (1), i.e. $[\![\kappa - A]\!] = [\![\kappa]\!] \cup \min([\![\overline{A}]\!], \preceq_\kappa)$. We have to distinguish between different cases:

case 1: In the first case we have $\kappa(A) = 0$, which implies $\gamma^- - \gamma^+ \leqslant -\kappa(\overline{A})$. There are two possibilities for a world ω to be an element of the minimal models of $\kappa^{\ominus}$, i.e. $\kappa^{\ominus}(\omega) = 0$:

For the case **1.A** we assume $\omega \models A$. Then we have $\kappa(\omega) = \kappa(\overline{A}) - (\gamma^+ - \gamma^-) \leqslant \kappa(\overline{A}) + \kappa(A) - \kappa(\overline{A}) = \kappa(A) \leqslant \kappa(\omega)$, which implies $\kappa(\omega) = \kappa(A)$ and therefore $\omega \in \min([\![A]\!], \preceq_\kappa)$ and

$\gamma^- - \gamma^+ = \kappa(A) - \kappa(\overline{A})$. In case **1.B** we assume $\omega \models \overline{A}$, which yields $\kappa(\omega) = \kappa(\overline{A})$, respectively $\omega \in \min(\llbracket\overline{A}\rrbracket, \preceq_\kappa)$. If we combine these two cases we get the following equation:

$$\llbracket\kappa^\ominus\rrbracket = \begin{cases} \min(\llbracket A\rrbracket, \preceq_\kappa) \cup \min(\llbracket\overline{A}\rrbracket, \preceq_\kappa) & \textit{if } \gamma^- - \gamma^+ = \kappa(A) - \kappa(\overline{A}) \\ \min(\llbracket\overline{A}\rrbracket, \preceq_\kappa) & \textit{if } \gamma^- - \gamma^+ < \kappa(A) - \kappa(\overline{A}) \end{cases}$$

With $\kappa(A) = 0$ we have that $\min(\llbracket A\rrbracket, \preceq_\kappa) \subseteq \llbracket\kappa\rrbracket$ holds. Assume that $\gamma^- - \gamma^+ < \kappa(A) - \kappa(\overline{A}) = -\kappa(\overline{A})$. In this case we have with (1) that $\min(\llbracket\overline{A}\rrbracket, \preceq_\kappa) = \min(\llbracket\overline{A}\rrbracket, \preceq_\kappa) \cup \llbracket\kappa\rrbracket$. From this we can derive that $\llbracket\kappa\rrbracket \subseteq \min(\llbracket\overline{A}\rrbracket, \preceq_\kappa)$ holds, which is a contradiction to the assumption that $\kappa(A) = 0$ holds. So the only possible case left is case 1.A. There we have $\gamma^- - \gamma^+ = \kappa(A) - \kappa(\overline{A}) = -\kappa(\overline{A})$ and with (1) we can conclude that $\min(\llbracket A\rrbracket, \preceq_\kappa) \cup \min(\llbracket\overline{A}\rrbracket, \preceq_\kappa) = \min(\llbracket\overline{A}\rrbracket, \preceq_\kappa) \cup \llbracket\kappa\rrbracket$ holds which leads to $\min(\llbracket A\rrbracket, \preceq_\kappa) \subseteq \llbracket\kappa\rrbracket$. This leads to the following equation for a c-contraction:

$$\kappa^\ominus(\omega) = \kappa(\omega) - \kappa(\overline{A}) + \begin{cases} \kappa(\overline{A}) - \kappa(A) & \textit{if } \omega \models A \\ 0 & \textit{if } \omega \models \overline{A} \end{cases} = \kappa(\omega) - \begin{cases} 0 & \textit{if } \omega \models A \\ \kappa(\overline{A}) & \textit{if } \omega \models \overline{A} \end{cases}$$

which corresponds to equation (15).
case 2: In the second case we have $\kappa(A) > 0$, which implies $\kappa(\overline{A}) = 0, \llbracket\kappa\rrbracket = \min(\llbracket\overline{A}\rrbracket, \preceq_\kappa)$. This gives us the following equation:

$$\kappa^\ominus(\omega) = \kappa(\omega) + \begin{cases} \gamma^+ - \gamma^- & \textit{if } \omega \models A \quad \text{with } \gamma^- - \gamma^+ \leqslant \kappa(A) \\ 0 & \textit{if } \omega \models \overline{A} \end{cases} \tag{16}$$

We now want to show that from these constraints we can derive that $\gamma^- - \gamma^+ < \kappa(A)$ holds. Assume we have $\gamma^- - \gamma^+ = \kappa(A)$. Then we have

$$\kappa^\ominus(\omega) = \kappa(\omega) + \begin{cases} -\kappa(A) & \textit{if } \omega \models A \\ 0 & \textit{if } \omega \models \overline{A} \end{cases} \tag{17}$$

We now want to look further into the minimal models of the contracted ranking function:

$$\kappa^\ominus(\omega) = 0 \Leftrightarrow \begin{cases} \omega \models A : \kappa(\omega) = \kappa(A) \Leftrightarrow \omega \in \min(\llbracket A\rrbracket, \preceq_\kappa) \\ \omega \models A : \kappa(\omega) = 0 \quad\quad \Leftrightarrow \omega \in \min(\llbracket\overline{A}\rrbracket, \preceq_\kappa) \end{cases}$$

which is equivalent to $\omega \in \min(\llbracket A\rrbracket, \preceq_\kappa) \cup \min(\llbracket\overline{A}\rrbracket, \preceq_\kappa)$. From this, together with (1), we can conclude that $\llbracket\kappa^\ominus\rrbracket = \min(\llbracket A\rrbracket, \preceq_\kappa) \cup \min(\llbracket\overline{A}\rrbracket, \preceq_\kappa) = \min(\llbracket\overline{A}\rrbracket, \preceq_\kappa) \cup \llbracket\kappa\rrbracket$ holds. We have $\kappa(A) > 0$ and hence $\llbracket\kappa\rrbracket = \min(\llbracket\overline{A}\rrbracket, \preceq_\kappa)$ which gives us $\min(\llbracket A\rrbracket, \preceq_\kappa) \cap \llbracket\kappa\rrbracket = \emptyset$ and so (1) cannot be fulfilled. So we can conclude that $\gamma^- - \gamma^+ < \kappa(A)$ which gives us equation (14). So (14) and (15) are the only possible equations for a c-contraction that fulfills Proposition 3. □

Note that (15) is a c-contraction in the spirit of the previously defined type α because the falsifying worlds of $\overline{A}$ are made more plausible whereas the ranks of the other worlds remain the same. However, (14) is a c-contraction in the spirit of type β, i.e., making A-worlds less plausible and not changing the ranks of the other worlds only if $\gamma^+ - \gamma^- \geqslant 0$.

6 Conditional preservation for iterated propositional contraction

We will now see that the approach of Konieczny and Pino Pérez [16], namely the axioms (IC1) - (IC4), can be derived from the principle of conditional preservation, together with (AGMes-1) -

(AGMes-7) and the following prerequisite that implements the intuition that under contraction by A, models of A should not be made more plausible:

$$\kappa \ominus A(\omega) \geqslant \kappa(\omega) \text{ for all } \omega \models A. \tag{18}$$

Theorem 9. *The axioms* ($PCP^{ocf}_{\ominus}$)*, (18), and* (AGMes-1) - (AGMes-7) *imply the axioms* (IC1) - (IC4) *in the context of OCFs. More precisely,* ($QPCP1^{sem}_{\circ}$) *implies the axioms* (IC1) *and* (IC2) *in the context of belief change by a proposition;* ($PCP^{ocf}_{\ominus}$) *together with (18) and* (AGMes-1) - (AGMes-7) *imply the axioms* (IC3) *and* (IC4)*.*

Proof. We will first show that ($\text{QPCP1}^{sem}_{\circ}$) $\Rightarrow$ (IC1) - (IC2) holds. Let $\circ$ be a belief change operator for propositions that fulfills ($\text{QPCP1}^{sem}_{\circ}$). For every proposition $A \in \mathcal{L}$ and $\omega_1, \omega_2 \in \Omega$ we know that $\sigma_{\mathcal{R}}(\omega_1) = \sigma_{\mathcal{R}}(\omega_2)$ implies $\omega_1 \preceq_{\Psi} \omega_2$ iff $\omega_1 \preceq_{\Psi \circ \mathcal{R}} \omega_2$ for $\mathcal{R} = \{(A|\top)\}$. From $\sigma_{\mathcal{R}}(\omega_1) = \sigma_{\mathcal{R}}(\omega_2)$ we can derive that either $\omega_1, \omega_2 \models A$ or $\omega_1, \omega_2 \models \overline{A}$ must hold which corresponds to the axioms (IC1) and (IC2). Since every ($\text{PCP}^{ocf}_{\circ}$) change is also a ($\text{QPCP1}^{sem}_{\circ}$) change we obtain the desired result.

Assume that $\ominus$ fulfills ($\text{PCP}^{ocf}_{\circ}$), (AGMes-1) - (AGMes-7) and (18). We will now prove that $\ominus$ fulfills (IC3) - (IC4) under the given presuppositions. According to Theorem 6 and 8, every contraction that satisfies ($\text{PCP}^{ocf}_{\ominus}$) and (AGMes-1) - (AGMes-7) has either the form (14) or the form (15). Let $\omega_1 \in [\![\overline{A}]\!], \omega_2 \in [\![A]\!]$ and $\kappa^{\ominus} = \kappa \ominus A$. For axiom (IC3) we have to show that $\omega_1 \prec_{\kappa} \omega_2 \Rightarrow \omega_1 \prec_{\kappa^{\ominus}} \omega_2$ holds, and for axiom (IC4) we have to show that $\omega_1 \preceq_{\kappa} \omega_2 \Rightarrow \omega_1 \preceq_{\kappa^{\ominus}} \omega_2$ holds. In the context of ranking functions we have $\omega_1 \preceq_{\kappa} \omega_2$ iff $\kappa(\omega_1) \leqslant \kappa(\omega_2)$. We first consider c-contractions of the form (14) and directly obtain $\kappa^{\ominus}(\omega_1) < \kappa^{\ominus}(\omega_2)$ iff $\kappa(\omega_1) < \kappa(\omega_2) + \gamma^+ - \gamma^-$. Since the prerequisite (18) ensures that $\gamma^+ - \gamma^- \geqslant 0$, we have if $\kappa(\omega_1) < \kappa(\omega_2)$ holds then also $\kappa^{\ominus}(\omega_1) < \kappa^{\ominus}(\omega_2)$. For every c-contraction of the form (15), $\kappa(\overline{A}) \geqslant 0$ yields that $\kappa(\omega_1) < \kappa(\omega_2)$ implies $\kappa(\omega_1) - \kappa(\overline{A}) < \kappa(\omega_2)$ and hence $\kappa^{\ominus}(\omega_1) < \kappa^{\ominus}(\omega_2)$. So if we have ($\text{PCP}^{ocf}_{\ominus}$), (18), and (AGMes-1) - (AGMes-7) we can derive the axiom (IC3). The proof of (IC4) is analogue. □

The following proposition is an immediate consequence of Theorem 9:

Proposition 10. *Every belief contraction for OCFs that fulfills* ($PCP^{ocf}_{\ominus}$)*, (18), and* (AGMes-1) - (AGMes-7) *also fulfills* (IC1) - (IC4)*.*

7 Conclusion and future work

In this paper we showed how the thoroughly axiomatized principle of conditional preservation of [14] can also be a leading paradigm for iterated contraction, and for iterated change operations in general. Based on this principle, we proposed a family of iterated contraction operators that follow the same schema as iterated revisions [14] but are shaped by constraints that implement success conditions and other goals of contraction. Our approach is compatible with the epistemic AGM postulates for contraction and further postulates for iterated contraction [16] which are similar to those of [4] for iterated revision. Even more, we were able to prove that the principle of conditional preservation together with natural constraints for contractions is powerful enough to bring forth all advanced postulates of [16]. Moreover, it also provides a schema for multiple iterated contraction by several propositions and / or conditionals. Elaborating such more general forms of iterated contraction is part of our ongoing work.

References

[1] C. Alchourrón, P. Gärdenfors, and D. Makinson. On the logic of theory change: Partial meet contraction and revision functions. *Journal of Symbolic Logic*, 50(2):510–530, 1985.

[2] R. Booth and J. Chandler. Extending the harper identity to iterated belief change. In S. Kambhampati, editor, *Proceedings of the Twenty-Fifth International Joint Conference on Artificial Intelligence, IJCAI 2016, New York, NY, USA, 9-15 July 2016*, pages 987–993. IJCAI/AAAI Press, 2016.

[3] T. Caridroit, S. Konieczny, and P. Marquis. Contraction in propositional logic. In S. Destercke and T. Denoeux, editors, *Symbolic and Quantitative Approaches to Reasoning with Uncertainty - 13th European Conference, ECSQARU 2015, Compiègne, France, July 15-17, 2015. Proceedings*, volume 9161 of *Lecture Notes in Computer Science*, pages 186–196. Springer, 2015.

[4] A. Darwiche and J. Pearl. On the logic of iterated belief revision. *Artificial Intelligence*, 89:1–29, 1997.

[5] B. DeFinetti. *Theory of Probability*, volume 1,2. John Wiley and Sons, New York, 1974.

[6] D. Dubois and H. Prade. Belief change and possibility theory. In P. Gärdenfors, editor, *Belief revision*, pages 142–182. Cambridge University Press, 1992.

[7] B. Fine and G. Rosenberger. *Algebraic Generalizations of Discrete Groups.* Dekker, New York, Basel, 1999.

[8] P. Gärdenfors. Belief revision and nonmonotonic logic: Two sides of the same coin? In *Proceedings European Conference on Artificial Intelligence, ECAI'92*, pages 768–773. Pitman Publishing, 1992.

[9] P. Gärdenfors and H. Rott. Belief revision. In D. Gabbay, C. Hogger, and J. Robinson, editors, *Handbook of Logic in Artificial Intelligence and Logic Programming*, pages 35–132. Oxford University Press, 1994.

[10] S. O. Hansson. Multiple and iterated contraction reduced to single-step single-sentence contraction. *Synthese*, 173(2):153–177, 2010.

[11] H. Katsuno and A. Mendelzon. Propositional knowledge base revision and minimal change. *Artificial Intelligence*, 52:263–294, 1991.

[12] G. Kern-Isberner. Following conditional structures of knowledge. In *KI-99: Advances in Artificial Intelligence, Proceedings of the 23rd Annual German Conference on Artificial Intelligence*, pages 125–136. Springer Lecture Notes in Artificial Intelligence LNAI 1701, 1999.

[13] G. Kern-Isberner. *Conditionals in nonmonotonic reasoning and belief revision.* Springer, Lecture Notes in Artificial Intelligence LNAI 2087, 2001.

[14] G. Kern-Isberner. A thorough axiomatization of a principle of conditional preservation in belief revision. *Annals of Mathematics and Artificial Intelligence*, 40(1-2):127–164, 2004.

[15] G. Kern-Isberner and D. Huvermann. What kind of independence do we need for multiple iterated belief change? *J. Applied Logic*, 22:91–119, 2017.

[16] S. Konieczny and R. Pino Pérez. On iterated contraction: syntactic characterization, representation theorem and limitations of the Levi identity. In *Scalable Uncertainty Management - 11th International Conference, SUM 2017, Granada, Spain, October 4-6, 2017, Proceedings*, volume 10564 of *Lecture Notes in Artificial Intelligence*. Springer, 2017.

[17] A. C. Nayak, R. Goebel, M. A. Orgun, and T. Pham. Taking Levi identity seriously: A plea for iterated belief contraction. In J. Lang, F. Lin, and J. Wang, editors, *Knowledge Science, Engineering and Management, First International Conference, KSEM 2006, Guilin, China, August 5-8, 2006, Proceedings*, volume 4092 of *Lecture Notes in Computer Science*, pages 305–317. Springer, 2006.

[18] R. Ramachandran, A. C. Nayak, and M. A. Orgun. Three approaches to iterated belief contraction. *J. Philosophical Logic*, 41(1):115–142, 2012.

[19] F. Ramsey. General propositions and causality. In R. Braithwaite, editor, *Foundations of Mathematics and other logical essays*, pages 237–257. Routledge and Kegan Paul, New York, 1950.

[20] W. Spohn. Ordinal conditional functions: a dynamic theory of epistemic states. In W. Harper and B. Skyrms, editors, *Causation in Decision, Belief Change, and Statistics, II*, pages 105–134. Kluwer Academic Publishers, 1988.

[21] W. Spohn. Measuring ranks via the complete laws of iterated contraction. In G. Bonanno, J. P. Delgrande, J. Lang, and H. Rott, editors, *Formal Models of Belief Change in Rational Agents, 26.08. - 30.08.2007*, volume 07351 of *Dagstuhl Seminar Proceedings*. Internationales Begegnungs- und Forschungszentrum fuer Informatik (IBFI), Schloss Dagstuhl, Germany, 2007.

[22] W. Spohn. *The Laws of Belief: Ranking Theory and Its Philosophical Applications*. Oxford University Press, 2012.