On the Quantity $m^2 - p^k$ Where $p^k m^2$ is an Odd Perfect Number

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Abstract: We prove that $m^2 - p^k$ is not a square, if $n = p^k m^2$ is an odd perfect number with special prime p, under the hypothesis that $\sigma(m^2)/p^k$ is a square. We are also able to prove the same assertion without this hypothesis. We also show that this hypothesis is incompatible with the set of assumptions

$$\left(m^2 - p^k \text{ is a power of two} \right) \wedge \left(p \text{ is a Fermat prime} \right).$$

We end by stating some conjectures.
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1   Introduction

Let $\sigma(x)$ denote the sum of the divisors of $x \in \mathbb{N}$, the set of positive integers. Denote the deficiency [13] of x by $D(x) = 2x - \sigma(x)$, and the sum of the aliquot divisors [14] of x by $s(x) = \sigma(x) - x$. Note that we have the identity $D(x) + s(x) = x$.

If a positive integer n is odd and $\sigma(n) = 2n$, then n is said to be an odd perfect number [17]. Euler proved that an odd perfect number, if one exists, must have the form $n = p^k m^2$, where p is the special prime satisfying $p \equiv k \equiv 1 \pmod 4$ and $\gcd(p, m) = 1$.

Descartes, Frenicle, and subsequently Sorli conjectured that $k = 1$ always holds [1]. Sorli conjectured $k = 1$ after testing large numbers with eight distinct prime factors for perfection [15]. Dris [7], and Dris and Tejada [12], call this conjecture as the Descartes-Frenicle-Sorli Conjecture, and derive conditions equivalent to $k = 1$.

Since m is odd, then $m^2 \equiv 1 \pmod 4$. Likewise, $p \equiv k \equiv 1 \pmod 4$, which implies that $p^k \equiv 1 \pmod 4$. It follows that $m^2 - p^k \equiv 0 \pmod 4$. Since

$$p^k < \frac{2m^2}{3}$$

(by a result of Dris [8]), we know a priori that

$$m^2 - p^k > \frac{p^k}{2}$$

so that we are sure that $m^2 - p^k > 0$. In particular, since $m^2 - p^k \equiv 0 \pmod 4$, we infer that $m^2 - p^k \geq 4$.

The index $i(p)$ of the odd perfect number $n = p^k m^2$ at the prime p is then equal to

$$i(p) := \frac{\sigma(m^2)}{p^k} = \frac{m^2}{\sigma(p^k)/2} = \frac{D(m^2)}{s(p^k)} = \frac{s(m^2)}{D(p^k)/2} = \gcd(m^2, \sigma(m^2)).$$

The term *index of an odd perfect number (at a certain prime)* was coined by Chen and Chen [4].

In this paper, we will refer continually to the following result by Broughan, et al. [(Lemma 8, page 7), [2]], which we will state without proof.

Lemma 1.1. *If $n = p^k m^2$ is an odd perfect number and $\sigma(m^2)/p^k$ is a square, then $k = 1$ holds.*

2 Summary

We now present a summary of our results in this section.

The first proposition allows us to rule out $m^2 - p^k = s^2$ (where $s \geq 2$), under the assumption that $\sigma(m^2)/p^k$ is a square.

Theorem 2.1. *If $n = p^k m^2$ is an odd perfect number and $\sigma(m^2)/p^k$ is a square, then $m^2 - p^k$ is not a square.*

In the second proposition, we remove the requirement that $\sigma(m^2)/p^k$ is a square and prove unconditionally that $m^2 - p^k$ is not a square, with some help from MSE user FredH (`https://math.stackexchange.com/users/82711`).

Theorem 2.2. *If $n = p^k m^2$ is an odd perfect number, then $m^2 - p^k$ is not a square.*

Finally, in the third proposition, we use the hypothesis that $\sigma(m^2)/p^k$ is a square to prove that $m^2 - p^k$ is not a power of two when p is a Fermat prime.

Theorem 2.3. *If $n = p^k m^2$ is an odd perfect number and $\sigma(m^2)/p^k$ is a square, then either $m^2 - p^k \neq 2^{2t+1}$ for integers $t \geq 1$ or p is not a Fermat prime.*

3   A Proof of Theorem 2.1

Suppose that $n = p^k m^2$ is an odd perfect number with special prime p, and that $m^2 - p^k = s^2$, for some $s \geq 2$.

Then

$$m^2 - s^2 = p^k = (m+s)(m-s)$$

so that we obtain

$$\begin{cases} p^{k-v} = m + s \\ p^v = m - s \end{cases}$$

where v is a positive integer satisfying $0 \leq v \leq (k-1)/2$. It follows that we have the system

$$\begin{cases} p^{k-v} + p^v = p^v(p^{k-2v} + 1) = 2m \\ p^{k-v} - p^v = p^v(p^{k-2v} - 1) = 2s \end{cases}$$

Since p is a prime satisfying $p \equiv 1 \pmod 4$ and $\gcd(p, m) = 1$, from the first equation it follows that $v = 0$, so that we obtain

$$\begin{cases} p^k + 1 = 2m \\ p^k - 1 = 2s \end{cases}$$

which yields

$$m = \frac{p^k + 1}{2} < p^k.$$

Lastly, note that the inequality $p < m$ has been proved by Brown [3], Dris [6], and Starni [16], so that we are faced with the inequality

$$p < m < p^k.$$

This implies that $k > 1$.

However, by assumption we have that $\sigma(m^2)/p^k$ is a square. This implies by Lemma 1.1 that $k = 1$, a clear contradiction.

This ends the proof of Theorem 2.1.

Remark 3.1. *The following shorter proof for Theorem 2.1 was communicated by a referee.*

First, since $\sigma(m^2)/p^k$ is a square by assumption, then $k = 1$ by Lemma 1.1.

Then $m^2 - p^k = m^2 - p$, and it is straightforward to show that $m^2 - p = s^2$ for $s \geq 2$ is impossible: This would imply $m = (p+1)/2$, which contradicts $p < m$.

4   A Proof of Theorem 2.2

The following proof is lifted verbatim from [10]:

Here's a way to finish the proof without appealing to any conjecture.

If $n = p^k m^2$ is a perfect number with $\gcd(p, m) = 1$, we have

$$\sigma(p^k)\sigma(m^2) = 2p^k m^2.$$

We know that $\sigma(p^k) = (p^{k+1}-1)/(p-1)$ and we have shown in Theorem 2.1 that $m = (p^k+1)/2$, so we can conclude that

$$2(p^{k+1} - 1)\sigma(m^2) = (p - 1)p^k(p^k + 1)^2. \qquad (*)$$

Consider the GCD of $p^{k+1} - 1$ with the right-hand side:

$$\gcd\left(p^{k+1} - 1, (p - 1)p^k(p^k + 1)^2\right) \leq (p - 1)\gcd(p^{k+1} - 1, p^k + 1)^2,$$

since p^k is coprime to $p^{k+1} - 1$.

Noticing that $p^{k+1} - 1 = p(p^k + 1) - (p + 1)$, we find $\gcd(p^{k+1} - 1, p^k + 1) = \gcd(p + 1, p^k + 1)$, which is $p + 1$ because k is odd.

Thus

$$\gcd(p^{k+1} - 1, (p - 1)p^k(p^k + 1)^2) \leq (p - 1)(p + 1)^2.$$

Since $k \equiv 1 \pmod 4$ and we have shown in Theorem 2.1 that $k > 1$, we have $k \geq 5$. If $(*)$ holds, the left-hand side of the inequality must be $p^{k+1} - 1$, which is then greater than p^5. But the right-hand side is less than p^4, so this is impossible.

This completes the proof of Theorem 2.2.

5   A Proof of Theorem 2.3

Suppose that $n = p^k m^2$ is an odd perfect number with special prime p, and that $\sigma(m^2)/p^k$ is a square. We show that the assumption

$$\left(m^2 - p^k = 2^{2t+1}, t \geq 1\right) \wedge \left(p \text{ is a Fermat prime}\right)$$

shall contradict Lemma 1.1.

(The following proof is adapted from the proof of Theorem 5 in [11].)

Assume to the contrary that

$$m^2 - p^k = 2^{2t+1}$$

for some integer $t \geq 1$, and that p is a Fermat prime. This means that

$$p = 2^r + 1$$

for some integer $r \geq 2$. Since p is a Fermat prime, we have $r = 2^l$, for some integer $l \geq 1$. In other words,

$$p = 2^{2^l} + 1$$

is a Fermat prime.

Now, note that it is trivial to prove that

$$3 \mid 2^{2^l-1} + 1 = \frac{p+1}{2}.$$

4

By assumption, $\sigma(m^2)/p^k$ is a square, which implies that $k = 1$. It follows that

$$m^2 - p = m^2 - (2^{2^l} + 1) = 2^{2t+1}$$

from which we get

$$m^2 - 2^{2^l} = 2^{2t+1} + 1,$$

which implies that

$$3 \mid \left(m^2 - 2^{2^l} \right).$$

This means that $3 \nmid m^2$, since $l \geq 1$ and $3 \nmid 2^{2^l}$.

But we know that $3 \mid (p+1)/2 \mid m^2$. This contradicts $3 \nmid m^2$.

This finishes the proof of Theorem 2.3.

Remark 5.1. *The divisibility constraint $(p+1)/2 \mid m^2$ is true in general since*

$$(p+1) = \sigma(p) \mid \sigma(p^k) \mid 2m^2$$

follows from $k \equiv 1 \pmod 4$, $\gcd(p^k, \sigma(p^k)) = 1$, and the equation

$$\sigma(p^k)\sigma(m^2) = \sigma(p^k m^2) = \sigma(n) = 2n = 2p^k m^2.$$

6 Concluding Remarks and Future Research

Actually, more stringent conditions on $m^2 - p^k$ can be derived when $\sigma(m^2)/p^k$ is a square. Since $\sigma(m^2)/p^k$ is always odd, and by assumption it is a square, then since $p \equiv k \equiv 1 \pmod 4$ holds, we know that $\sigma(m^2) \equiv 1 \pmod 4$ also holds. This last congruence is known to hold if and only if $p \equiv k \pmod 8$. ([9], [5]) Since by assumption $\sigma(m^2)/p^k$ is a square, we obtain $k = 1$ by Lemma 1.1. In particular, we know that $p^k \equiv 1 \pmod 8$. But we also know that m is odd. Therefore, we infer that $m^2 \equiv 1 \pmod 8$. It follows that $m^2 - p^k \equiv 0 \pmod 8$.

What follows is an elementary attempt to rule out $m^2 - p^k = 8$:

Lemma 6.1. *If $n = p^k m^2$ is an odd perfect number with special prime p and $\sigma(m^2)/p^k$ is a square, then $m^2 - p^k \neq 8$.*

Proof. Let $n = p^k m^2$ be an odd perfect number with special prime p. Suppose that $\sigma(m^2)/p^k$ is a square.

Assume to the contrary that $m^2 - p^k = 8$. Subtract 9 from both sides, then transfer p^k to the right-hand side:

$$m^2 - 9 = p^k - 1$$

$$(m+3)(m-3) = p^k - 1$$

This last equation implies that, in general, we have the divisibility constraint

$$(m+3) \mid (p^k - 1).$$

This divisibility constraint then implies that

$$(m + 3) \leq (p^k - 1)$$

from which we obtain

$$m < m + 4 \leq p^k.$$

Lastly, note that the inequality $p < m$ has been proved by Brown [3], Dris [6], and Starni [16], so that we are faced with the inequality

$$p < m < p^k.$$

But this contradicts Lemma 1.1, so we are done.

This ends the proof of Lemma 6.1. □

We end this section with the following conjectures, which we leave for other researchers to investigate.

Conjecture 6.2. *If $n = p^k m^2$ is an odd perfect number and $\sigma(m^2)/p^k$ is a square, then $m^2 - p^k$ is not a cube.*

Conjecture 6.3. *If $n = p^k m^2$ is an odd perfect number, then $m^2 - p^k$ is not a cube.*

Acknowledgements

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