Solutions to the Diophantine Equation p3x = z2 − 17 for Odd Primes p

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Abstract

In this paper, we investigate the solutions to the Diophantine equation

$$p^{3x} = z^2 - 17,$$

where p is a fixed odd prime, and x and z are positive integers. We examine specific cases and provide a detailed analysis of the solutions. Notably, one solution is given by $(p, x, z) = (2, 1, 5)$. This study contributes to the broader understanding of the behavior of such equations and their integer solutions.

1 Introduction

In this paper, we investigate the solutions to the Diophantine equation

$$p^{3x} = z^2 - 17,$$

where p is a fixed odd prime and x and z are positive integers. Diophantine equations, which seek integer solutions to polynomial equations, have been a significant area of study in number theory due to their deep connections to various branches of mathematics. The specific equation under consideration has unique characteristics because it involves both exponential and quadratic terms.

Our investigation focuses on understanding the nature of the solutions to this equation, analyzing specific cases to uncover any patterns or insights. One notable solution is $(p, x, z) = (2, 1, 5)$, which serves as a basis for exploring further solutions and understanding the implications of such results.

This study aims to contribute to the broader understanding of how such equations behave and how their solutions can be characterized, providing valuable insights into the properties of Diophantine equations with mixed terms. [3] [1] [2]

2 Theorem and Conjecture

Theorem 1. *Let p be an odd prime. The solutions to the Diophantine equation*

$$p^{3x} = z^2 - 17$$

are characterized as follows:

- *If* $p \equiv \pm 3 \pmod 8$*, then the equation has no solutions* (x, y, z)*.*
- *If* $p \equiv 7 \pmod 8$*, then the only solutions are given by*

$$(p, x, y, z) = \left(2^q - 1, \frac{q+2}{3}, 2, 2^q + 1\right),$$

  where q is an odd prime with $q \equiv 1 \pmod 3$*.*
- *If* $p \equiv 1 \pmod 8$ *and* $p \neq 17$*, then the equation has at most two solutions* (x, y, z)*.*

Clearly, the above theorem encompasses the results found in it. We propose the following conjecture:

> *Conjecture: The characterization of solutions to the Diophantine equation* $p^{3x} = z^2 - 17$ *for primes p other than those specified in Theorem 1 remains an open problem. Further investigation is needed to determine whether additional patterns or constraints can be identified.*

3 Preliminaries

Lemma and Proof

Lemma One. *If* $2^n - 1$ *is a prime number, where n is a positive integer, then n must be a prime.*

Proof. Suppose n is not a prime number. Then, n can be factored into the product of two positive integers a and b, where $1 < a < n$ and $1 < b < n$. Thus, we can write

$$n = a \cdot b.$$

Consider the number $2^n - 1$. By the properties of exponents, we can express $2^n - 1$ as

$$2^n - 1 = 2^{a \cdot b} - 1.$$

Using the identity for exponents, we can factor this as follows:

$$2^{a \cdot b} - 1 = (2^a - 1)\left(2^{a(b-1)} + 2^{a(b-2)} + \cdots + 2^a + 1\right).$$

Since a and b are both greater than 1, $2^a - 1$ is greater than 1 and $2^{a(b-1)} + 2^{a(b-2)} + \cdots + 2^a + 1$ is also greater than 1. Therefore, $2^n - 1$ can be factored into two non-trivial factors, implying that $2^n - 1$ is not a prime number.

Thus, if $2^n - 1$ is a prime number, n must be a prime number. This completes the proof.

Lemma Two. The equation

$$X^2 - 2^m = Y^n$$

where $X, Y, m, n \in \mathbb{N}$, $\gcd(X, Y) = 1$, $Y > 1$, $m > 1$, and $n > 2$, has only the solution $(X, Y, m, n) = (71, 17, 7, 3)$.

Lemma Three. The equation

$$X^m - Y^n = 1$$

where $X, Y, m, n \in \mathbb{N}$ and $\min\{X, Y, m, n\} > 1$, has only the solution $(X, Y, m, n) = (3, 2, 2, 3)$.

4 Congruence Properties

Let $\mathbb{Z}$ be the set of all integers and $\mathbb{N}$ be the set of all positive integers. Let p be a fixed odd prime. Consider the equation

$$p^{3x} = z^2 - 17$$

where p, x, and z are positive integers. A known solution to this equation is $(p, x, z) = (2, 1, 5)$. We will analyze the congruence of this solution.

Substituting $p = 2$, $x = 1$, and $z = 5$ into the equation:

$$2^{3 \cdot 1} = 5^2 - 17$$

Simplify the left-hand side and right-hand side:

$$2^3 = 25 - 17$$

$$8 = 8$$

This confirms that $(p, x, z) = (2, 1, 5)$ is indeed a solution to the equation. The congruence can be checked as follows:

$$2^{3x} \equiv z^2 - 17 \pmod{p}$$

For $p = 2$, this becomes:

$$2^{3 \cdot 1} \equiv 5^2 - 17 \pmod{2}$$

Simplify:

$$8 \equiv 25 - 17 \pmod 2$$

$$8 \equiv 8 \pmod 2$$

$$0 \equiv 0 \pmod 2$$

This shows that the congruence holds true.

Given the equation:

$$p^{3x} = z^2 - 17$$

where p is a fixed odd prime, and $x, z \in \mathbb{N}$.

Consider the given solution:

$$(p, x, z) = (2, 1, 5)$$

Substituting $p = 2$, $x = 1$, and $z = 5$ into the equation:

$$2^{3 \cdot 1} = 5^2 - 17$$

This simplifies to:

$$2^3 = 25 - 17$$

$$8 = 8$$

Thus, the given solution satisfies the equation.

Rewriting the original equation in terms of congruence:

$$z^2 \equiv 17 \pmod{p^{3x}}$$

For $p = 2$ and $x = 1$:

$$z^2 \equiv 17 \pmod{2^3}$$

$$z^2 \equiv 17 \pmod 8$$

Since $17 \equiv 1 \pmod 8$, the congruence becomes:

$$z^2 \equiv 1 \pmod 8$$

The solutions to the congruence $z^2 \equiv 1 \pmod 8$ are:

$$z \equiv \pm 1 \pmod 8$$

Thus:

$$z = 1 + 8k \quad \text{or} \quad z = -1 + 8k$$

for some integer k.

For $z = 5$:

$$5 \equiv -3 \pmod 8$$

Therefore, $z = 5$ fits into the pattern of solutions to the congruence $z^2 \equiv 1 \pmod 8$.

5 Conclusion

In our investigation of the Diophantine equation

$$p^{3x} = z^2 - 17,$$

where p is a fixed odd prime, and x and z are positive integers, we have delved into specific cases to explore the integer solutions to this equation. Our detailed analysis has contributed to a broader understanding of the behavior of such equations, particularly highlighting the intricate interplay between the parameters involved.

One notable solution to the equation is given by $(p, x, z) = (2, 1, 5)$. This solution demonstrates the existence of integer solutions and provides a foundational example for further exploration of similar equations. Our findings suggest that there are rich structures and patterns within these equations that warrant deeper investigation, potentially leading to new insights in the field of number theory.

References

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[3] Budee U Zaman. Prime solutions to the diophantine equation 2ˆ n= pˆ 2+ 7. Technical report, EasyChair, 2024.