Output-Reference Finite-Time Bounded Tracking Control of Linear Systems with Disturbance Generated by an Exosystem

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***Abstract*—In present paper the tracking control problem of linear discrete-time systems with disturbance in a finite time interval is considered. We constructed a new system so that the problem can be turned into finite-time boundedness(FTB) problem of the output vector. Then a finite-time state feedback controller of the system is derived by designing the state feedback controller for the new system via the linear matrix inequality(LMI) approach. Based on this result, a finite-time bounded tracking controller of the original system is obtained. Finally, a numerical example is presented to illustrate the feasibility of the controller.**

***Index Terms*—discrete-time, finite-time bounded tracking Control, disturbance, state feedback controller, LMI approach**

I. Introduction

Many researchers have studied asymptotic stability of dynamical systems since its emergence at the end of the 19^th with the fundamental theorem of Lyapunov in [1]. But there is an important property which concerns this kind of stability: finite-time stability. Now we call a system is to be finite-time stable(FTS) if its state dose not exceeds some bounds during a fixing time-interval. In many practical applications, for example, in robot control systems, communication network systems, or biochemical reaction systems, people are more interested in what happens over a finite-time interval rather than the asymptotical property in a infinite time. To discuss this transient performance, Dorato [2] firstly defined finite-time stability for linear deterministic systems. Recently, significant contributions have been given in this field, especially in the case of linear systems.

With the development of linear matrix inequality(LMI) theory, the researches on FTS yielded fruitful results. In [3-4], Amato et al. extended the concept of FTS to the linear continuous-time system with external disturbances and gave the concept of finite-time boundedness(FTB), and the discrete-time system was also investigated in [5-6]. The further researches we refer the readers to [7-10].

Tracking control is one of the dynamic problem and most significant topic, both in control field and practice. In general, the tracking problem is very much difficult in contrast to stability problem[11-12]. One of the most important problem about tracking control is the controller design[13-14]. Tracking controller is generally regarded as a device to control the response of a system to track a desired trajectory in an exact manner. According to the desired values, there are two categories for tracking problem[15]: output tracking problem and state tracking problem. The tracking control design is already proposed in the continuous-time domain[16]. Now, many controllers are designed in discrete-time domain. In contrast to continuous-time, discrete-time approach has many advantages(e.g. better implementation, adaptable). It can handle a very much wider class of control laws over continuous-time domain[17-18].

Throughout this paper, $\mathbb{R}^n$ represents $n-$dimensional real Euclidean space. The matrix P^T represents the transposed matrix of P. For a symmetric matrix, $P = P^T > 0$ means that matrix is positive definite. $P > Q$ means that $P - Q > 0$. $\lambda_{max}(A)(\lambda_{min}(A))$ denotes the maximal (or minimal) eigenvalue of a real symmetric matrix A. By diag$(\cdots)$ we denotes a block-diagonal matrix.

The rest of this paper is organized as follows. In Section II, we provide the considered system description and some definitions and results. In section III, we constructed a new system according to the orginal system so that the problem can be turned into FTB problem of the output vector. Section IV is the main part of this paper. A finite-time state feedback controller of the system is derived by designing the state feedback controller for the new system via the LMI approach. Based on this, a finite-time bounded tracking controller of the original system is obtained. In section V simulation results provided to illustrate the feasibility of the controller.

II. Definitions and Preliminary Results

The paper is concerned with the trajectory tracking control problem for a class of linear discrete-time systems, whose dynamics can be described by

$$x(k+1) = Ax(k) + Ew(k), \quad x(0) = x_0 \tag{1}$$

where $x(k) \in \mathbb{R}^n$ is the state vector, $w(k) \in \mathbb{R}^p$ is the disturbance generated by the exosystem

$$w(k+1) = Sw(k), \quad w(0) = w_0 \tag{2}$$

and all matrices are bounded of compatible dimensions.

The finite-time bounded(FTB) of system with exosystem was described as follows: system (1)-(2) is said to be finite-time bounded with respect to $(\delta_x, \delta_w, \varepsilon, R, N)$, where $N \geq 1, \delta_w > 0, \varepsilon > 0, \delta_x > 0$, and $R > 0$, if

$$x^T(0)Rx(0) \leq \delta^2,\ w^T(0)w(0) \leq \delta_w^2 \ \Rightarrow \ x^T(k)Rx(k) \leq \varepsilon^2,$$

$\forall k \in \{1, 2, \cdots, N\}$.

For convenience, hereinafter, the state vector of system (1)-(2)is also said to be finite-time bounded with respect to $(\delta_x, \delta_w, \varepsilon, R, N)$. The object of this paper is to discuss the finite-time bounded tracking of system (1) with exosystem (2). Now we first propose a definition of finite-time bounded tracking.

Consider the discrete-time system

$$\begin{aligned} x(k+1) &= Ax(k) + Ew(k) \\ w(k+1) &= Sw(k) \\ y(k) &= Cx(k), \end{aligned} \tag{3}$$

where $x(k) \in \mathbb{R}^n, w(k) \in \mathbb{R}^p$ and $y(k) \in \mathbb{R}^q$ are the state vector, the disturbance vector, and the output vector of the system, respectively. $A \in \mathbb{R}^{n\times n}, E \in \mathbb{R}^{n\times p}, S \in \mathbb{R}^{p\times p}$ and $C \in \mathbb{R}^{q\times n}$ are known constant matrices.

In some practical problems, it is hoped that the output of system (3) is always located in a neighborhood of a reference signal under some certain conditions. This kind of problem is referred to as ”finite-time bounded tracking problem”. Let the reference signal be $r(k) \in \mathbb{R}^q$ generated by the following system

$$r(k+1) = Mr(k), \qquad r(0) = r_0. \tag{4}$$

The error signal $e(k)$ is defined as

$$e(k) = y(k) - r(k). \tag{5}$$

The concept mentioned above can be described by the following definition.

Definition 1 System (3) is finite-time bounded tracking of the reference signal $r(k)$ with respect to $(\delta_e, \delta_w, \varepsilon, R, N)$, where $N \geq 1, \delta_x > 0, \delta_w > 0, \varepsilon > 0$, and $R > 0$, if

$$e^T(0)Re(0) \leq \delta_e^2,\ w^T(0)w(0) \leq \delta_w^2 \ \Rightarrow \ e^T(k)Re(k) \leq \varepsilon^2,$$

$\forall k \in \{1, 2, \cdots, N\}$.

To deduce an LMI feasibility problem, the Schur complement lemma is also needed.

Lemma 1 Symmetric matrix $\begin{pmatrix} X_{11} & X_{12} \\ X_{12} & X_{22} \end{pmatrix} < 0$ if and only if one of the following two conditions is satisfied:

$$\begin{aligned} &(1)\ X_{11} < 0, \quad S_{22} - S_{12}^T S_{11}^{-1} S_{12} < 0. \\ &(2)\ X_{22} < 0, \quad S_{11} - S_{12} S_{22}^{-1} S_{12}^T < 0. \end{aligned}$$

III. CONSTRUCTION OF THE ERROR SYSTEM

Now consider the following discrete-time system with disturbance generated by an exosystem

$$\begin{aligned} x(k+1) &= Ax(k) + Ew(k) + Bu(k) \\ w(k+1) &= Sw(k) \\ y(k) &= Cx(k), \end{aligned} \tag{6}$$

where $x(k) \in \mathbb{R}^n, u(k) \in \mathbb{R}^m, w(k) \in \mathbb{R}^p$ and $y(k) \in \mathbb{R}^q$ are the state vector, the input vector, the disturbance vector, and the output vector of the system, respectively. $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}, E \in \mathbb{R}^{n\times p}, S \in \mathbb{R}^{p\times p}$ and $C \in \mathbb{R}^{q\times n}$ are known constant matrices.

The difference operator Δ of vector and matrix is defined as

$$\Delta\nu(k+1) = \nu(k+1) - \nu(k).$$

Taking the operator Δ on both sides of the first equation of (6), one can follows

$$\Delta x(k+1) = A\Delta x(k) + E\Delta w(k) + B\Delta u(k). \tag{7}$$

Applying Δ to $e(k+1) = y(k+1) - r(k+1)$ and noting that $\Delta e(k+1) = e(k+1) - e(k)$, we have

$$\begin{aligned} e(k+1) &= e(k) + \Delta y(k+1) - \Delta r(k+1) \\ &= e(k) + C\Delta x(k) - \Delta r(k+1) \\ &= e(k) + CA\Delta x(k) + CB\Delta u(k) \\ &\quad + CE\Delta w(k) - \Delta r(k+1). \end{aligned} \tag{8}$$

Introduce the formal state vector

$$X(k) = \begin{pmatrix} e(k) \\ \Delta x(k) \end{pmatrix}$$

and matrices

$$\bar{A} = \begin{pmatrix} I & CA \\ 0 & A \end{pmatrix},\ \bar{B} = \begin{pmatrix} CB \\ B \end{pmatrix},\ \bar{E} = \begin{pmatrix} CE \\ E \end{pmatrix},\ \bar{R} = \begin{pmatrix} -I \\ 0 \end{pmatrix},$$

then by (7) and (8) we get

$$X(k+1) = \bar{A}X(k) + \bar{E}\Delta w(k) + \bar{B}\Delta u(k) + \bar{R}\Delta r(k+1).$$

Combine the terms $\Delta w(k)$ and $\Delta r(k+1)$ in above equation and take them as the external disturbance, we have

$$X(k+1) = \bar{A}X(k) + \bar{B}\Delta u(k) + \begin{pmatrix} \bar{E} & \bar{R} \end{pmatrix} \begin{pmatrix} \Delta w(k) \\ \Delta r(k+1) \end{pmatrix}. \tag{9}$$

Let $W(k) = \begin{pmatrix} \Delta w(k) \\ \Delta r(k+1) \end{pmatrix}, \tilde{E} = \begin{pmatrix} \bar{E} & \bar{R} \end{pmatrix}, \bar{S} = \begin{pmatrix} S & 0 \\ 0 & M \end{pmatrix}$. Define a new exosystem and output as follows, respectively,

$$W(k+1) = \bar{S}W(k), \tag{10}$$

$$Y(k) = \bar{C}X(k), \tag{11}$$

where $k = 1, 2, \cdots, N, \bar{C} = \begin{pmatrix} I & 0 \end{pmatrix}$. Combine (9)-(11) one can get a new system with a new exosystem

$$\begin{aligned} X(k+1) &= \bar{A}X(k) + \tilde{E}W(k) + \bar{B}\Delta u(k) \\ W(k+1) &= \bar{S}W(k) \\ Y(k) &= \bar{C}X(k) \end{aligned} \tag{12}$$

Now we set a formal reference signal $R(k) = 0, k = 1, 2, \cdots, N$. Obviously, $e(k) = Y(k) - R(k)$. Thus, an error system can be gotten as follows:

$$\begin{aligned} X(k+1) &= \bar{A}X(k) + \tilde{E}W(k) + \bar{B}\Delta u(k) \\ W(k+1) &= \bar{S}W(k) \\ Y(k) &= \bar{C}X(k) \\ e(k) &= Y(k) - R(k) \end{aligned} \tag{13}$$

IV. Design of the Controller

Consider the following state feedback controller

$$\Delta u(k) = FX(k),$$

where $F = \begin{pmatrix} F_e & F_x \end{pmatrix}$ will be determined later. By (13) it follows

$$\begin{aligned} X(k+1) &= (\bar{A} + \bar{B}F)X(k) + \tilde{E}W(k) \\ W(k+1) &= \bar{S}W(k) \\ Y(k) &= \bar{C}X(k) \\ e(k) &= Y(k) - R(k) \end{aligned} \tag{14}$$

Let $r^T(0)r(0) = \delta_r^2$. Then

$$\begin{aligned} W^T(1)W(1) &= \begin{pmatrix} \Delta w^T(1) & \Delta r^T(2) \end{pmatrix} \begin{pmatrix} \Delta w(1) \\ \Delta r(2) \end{pmatrix} \\ &= \Delta w^T(1)\Delta w(1) + \Delta r^T(2)\Delta r(2) \\ &\le 2(\lambda_{max}(S^TS) + 1)\delta_w^2 + 2(\lambda_{max}^2(M^TM) \\ &\quad + \lambda_{max}(M^TM) + 1)\delta_r^2. \end{aligned}$$

Let $\lambda_0 = \max\{\lambda_{max}(S^TS), \lambda_{max}(M^TM)\}$, and $\delta_W^2 = 2(\lambda_0 + 1)\delta_w^2 + 2(\lambda_0^2 + \lambda_0 + 1)\delta_r^2$, it follows that

$$W^T(1)W(1) \le \delta_W^2.$$

Theorem 1. The closed-loop system (14) is finite-time bounded tracking of the reference signal $R(k)$ with respect to $(\delta_e, \delta_W, \varepsilon, R, N)$, if for a given scalar $\gamma > 1$, there exist matrices $P_1 > 0, P_2 > 0$ and scalars $\lambda_1 > 0, \lambda_2 > 0$ such that

$$\begin{pmatrix} (\bar{A} + \bar{B}F)^T\bar{C}^TP_1\bar{C}(\bar{A} + \bar{B}F) - \gamma\bar{C}^TP_1\bar{C} & (\bar{A} + \bar{B}F)^T\bar{C}^TP_1\bar{C}\tilde{E} \\ \tilde{E}^T\bar{C}^TP_1\bar{C}(\bar{A} + \bar{B}F) & \tilde{E}^T\bar{C}^TP_1\bar{C}\tilde{E} - \gamma P_2 \end{pmatrix} < 0 \tag{15}$$

$$R < P_1 < \lambda_1 R, \tag{16}$$

$$0 < P_2 < \lambda_2 I, \tag{17}$$

$$\lambda_1\delta_e^2 + \lambda_2\sum_{i=1}^{N}\lambda_0^i\delta_W^2 < \frac{\varepsilon^2}{\gamma^{N-1}}. \tag{18}$$

Proof Construct the Lyapunov function as follows:

$$V(e(k)) = e^T(k)P_1e(k).$$

Then

$$\begin{aligned} V(e(k+1)) &= e^T(k+1)P_1e(k+1) \\ &= (\bar{C}X(k+1))^TP_1(\bar{C}X(k+1)) \\ &= (\bar{C}((\bar{A} + \bar{B}F)X(k) + \tilde{E}\Delta W(k)))^T \\ &\quad \cdot P_1(\bar{C}((\bar{A} + \bar{B}F)X(k) + \tilde{E}\Delta W(k))) \\ &= (\bar{C}(\bar{A} + \bar{B}F)X(k) + \bar{C}\tilde{E}\Delta W(k))^T \\ &\quad \cdot P_1(\bar{C}(\bar{A} + \bar{B}F)X(k) + \bar{C}\tilde{E}\Delta W(k)) \\ &= \begin{pmatrix} X^T(k) & W^T(k) \end{pmatrix} \begin{pmatrix} (\bar{A} + \bar{B}F)^T\bar{C}^T \\ \tilde{E}^T\bar{C}^T \end{pmatrix} \\ &\quad \cdot P_1 \begin{pmatrix} \bar{C}(\bar{A} + \bar{B}F) & \bar{C}\tilde{E} \end{pmatrix} \begin{pmatrix} X(k) \\ W(k) \end{pmatrix} \\ &= \begin{pmatrix} X^T(k) & W^T(k) \end{pmatrix} \\ &\quad \cdot \begin{pmatrix} (\bar{A} + \bar{B}F)^T\bar{C}^TP_1\bar{C}(\bar{A} + \bar{B}F) & (\bar{A} + \bar{B}F)^T\bar{C}^TP_1\bar{C}\tilde{E} \\ \tilde{E}^T\bar{C}^TP_1\bar{C}(\bar{A} + \bar{B}F) & \tilde{E}^T\bar{C}^TP_1\bar{C}\tilde{E} \end{pmatrix} \\ &\quad \cdot \begin{pmatrix} X(k) \\ W(k) \end{pmatrix}. \end{aligned}$$

By (15) one can obtain

$$\begin{aligned} V(e(k+1)) &\le \gamma V(e(k)) + \gamma W^T(k)P_2W(k) \\ &\le \gamma V(e(k)) + \lambda_{max}(P_2)\gamma W^T(k)W(k) \end{aligned} \tag{19}$$

Using recurrent calculation with (19) we have

$$\begin{aligned} V(e(k)) &\le \gamma V(e(k-1)) + \lambda_{max}(P_2)\gamma W^T(k-1)W(k-1) \\ &\le \gamma^2V(e(k-2)) \\ &\quad + \lambda_{max}(P_2)\gamma^2W^T(k-2)W(k-2) \\ &\quad + \lambda_{max}(P_2)\gamma W^T(k-1)W(k-1) \\ &\le \cdots \\ &\le \gamma^{k-1}V(e(1)) \\ &\quad + \lambda_{max}(P_2)\sum_{i=1}^{k-1}\gamma^iW^T(k-i)W(k-i). \end{aligned} \tag{20}$$

Taking $\gamma > 1$ into account, one can get from (20) that

$$\begin{aligned} V(e(k)) &\le \gamma^{k-1}[V(e(1)) \\ &\quad + \lambda_{max}(P_2)\sum_{i=1}^{k-1}W^T(k-i)W(k-i)]. \end{aligned} \tag{21}$$

Since for all $k \in \{1, 2, \cdots, N\}$ (21) holds, we have

$$\begin{aligned} V(e(k)) &\le \gamma^{N-1}[V(e(1)) \\ &\quad + \lambda_{max}(P_2)\sum_{i=0}^{N-1}W^T(N-i)W(N-i)]. \end{aligned} \tag{22}$$

Note that $\lambda_{max}(\bar{S}^T\bar{S}) = \lambda_0$, so

$$\begin{aligned}\sum_{i=0}^{N-1} W^T(N-i)W(N-i) =& \sum_{i=1}^{N} W^T(i)W(i)\\ \leq& \sum_{i=1}^{N} \lambda_0^i W^T(1)W(1)\\ \leq& \sum_{i=1}^{N} \lambda_0^i \delta_W. \end{aligned}\tag{23}$$

Denoting $P_1(R) = R^{-\frac{1}{2}}P_1R^{-\frac{1}{2}}$, then

$$\begin{aligned}V(e(1)) =& e(1)^T P_1 e(1)\\ \leq& \lambda_{max}(P_1(R))e(1)^T Re(1).\end{aligned}\tag{24}$$

Substituting (23) and (24) into (22) yields the following estimation:

$$V(e(k)) \leq \gamma^{N-1} \cdot \left(\lambda_{max}(P_1(R))e(1)^T Re(1) + \lambda_{max}(P_2)\sum_{i=1}^{N}\lambda_0^i\delta_W\right).\tag{25}$$

Since condition (16) is equivalent to $I < R^{-\frac{1}{2}}P_1R^{-\frac{1}{2}} < \lambda_1 I$, i.e. $I < P_1(R) < \lambda_1 I$, it follows

$$1 < \lambda_{min}(P_1(R)) \leq \lambda_{max}(P_1(R)) < \lambda_1.\tag{26}$$

In addition, condition (17) implies

$$0 < \lambda_{min}(P_2) \leq \lambda_{max}(P_2) < \lambda_2.$$

Thus, if (16) and (17) hold, it can be followed from (25) that

$$V(e(k)) \leq \gamma^{N-1}\left(\lambda_1\delta_e^2 + \lambda_2\sum_{i=1}^{N}\lambda_0^i\delta_W\right).\tag{27}$$

By (26) one can get

$$\begin{aligned}V(e(k)) =& e(k)^T P_1 e(k)\\ \geq& \lambda_{min}(P_1(R))e(k)^T Re(k)\\ \geq& e(k)^T Re(k).\end{aligned}$$

Thus, by (18) we have

$$\begin{aligned}e(k)^T Re(k) \leq& \gamma^{N-1}\left(\lambda_1\delta_e^2 + \lambda_2\sum_{i=1}^{N}\lambda_0^i\delta_W\right)\\ <& \varepsilon^2.\end{aligned}\tag{28}$$

The proof is completed.

Next we will give a sufficient condition that guarantees the system (6) is finite-time bounded tracking of the reference signal $r(k)$ with respect to $(\delta_e, \delta_w, \varepsilon, R, N)$

Theorem 2. The closed-loop system (6) is finite-time bounded tracking of the reference signal $r(k)$ with respect to $(\delta_e, \delta_w, \varepsilon, R, N)$, if for a given scalar $\gamma > 1$, there exist matrices $Q_1 > 0, P_2 > 0$ and scalars $\lambda_1' > 0, \lambda_2 > 0$ such that

$$\begin{pmatrix} -\gamma Q_1 & 0 & 0 & Q_1 + L^TB^TC^T\\ 0 & 0 & 0 & A^TC^T + F_x^TB^TC^T\\ 0 & 0 & -\gamma P_2 & H^T\\ Q_1 + CBL & CA + CBF_x & H & -Q_1 \end{pmatrix} < 0,\tag{29}$$

$$\lambda_1' R^{-1} < Q_1 < R^{-1},\tag{30}$$

$$0 < P_2 < \lambda_2 I,\tag{31}$$

$$\begin{pmatrix}\lambda_2 d^2 - \frac{\varepsilon^2}{\gamma^{N-1}} & \delta_e\\ \delta_e & -\lambda_1'\end{pmatrix} < 0.\tag{32}$$

where $H = \begin{pmatrix} CE & -I\end{pmatrix}$, $d = \sum_{i=1}^{N}\lambda_0^i\delta_W^2$, $\delta_W^2 = 2(\lambda_0 + 1)\delta_w^2 + 2(\lambda_0^2 + \lambda_0 + 1)\delta_r^2$, $\lambda_0 = \max\{\lambda_{max}(S^TS),\ \lambda_{max}(M^TM)\}$. In this case the controller is $\Delta u(k) = F_e e(k) + F_x\Delta x(k)$ with $F_e = LQ_1^{-1}$.

Proof Let

$$P_2 = \begin{pmatrix} P_{11} & P_{12}\\ P_{21} & P_{22}\end{pmatrix},\ U = I + CBF_e,\ V = CA + CBF_x.$$

Note that

$$\bar{A} = \begin{pmatrix} I & CA\\ 0 & A\end{pmatrix},\ \bar{B} = \begin{pmatrix} CB\\ B\end{pmatrix},\ \bar{E} = \begin{pmatrix} CE\\ E\end{pmatrix},\ \bar{R} = \begin{pmatrix} -I\\ 0\end{pmatrix},$$

$$\tilde{E} = \begin{pmatrix}\bar{E} & \bar{R}\end{pmatrix},\ \bar{C} = \begin{pmatrix} I & 0\end{pmatrix},\ F = \begin{pmatrix} F_e & F_x\end{pmatrix}.$$

Plug above equations into the following inequality:

$$\begin{pmatrix} (\bar{A}+\bar{B}F)^T\bar{C}^TP_1\bar{C}(\bar{A}+\bar{B}F) - \gamma\bar{C}^TP_1\bar{C} & (\bar{A}+\bar{B}F)^T\bar{C}^TP_1\bar{C}\tilde{E}\\ \tilde{E}^T\bar{C}^TP_1\bar{C}(\bar{A}+\bar{B}F) & \tilde{E}^T\bar{C}^TP_1\bar{C}\tilde{E} - \gamma P_2\end{pmatrix} < 0,\tag{33}$$

we have

$$\begin{pmatrix} U^TQ_1^{-1}U - \gamma Q_1^{-1} & U^TQ_1^{-1}V & U^TQ_1^{-1}CE & -U^TQ_1^{-1}\\ V^TQ_1^{-1}U & V^TQ_1^{-1}V & V^TQ_1^{-1}CE & -V^TQ_1^{-1}\\ (CE)^TQ_1^{-1}U & (CE)^TQ_1^{-1}V & (CE)^TQ_1^{-1}CE - \gamma P_{11} & -(CE)^TQ_1^{-1} - \gamma P_{12}\\ -Q_1^{-1}U & -Q_1^{-1}V & -Q_1^{-1}CE - \gamma P_{21} & Q_1^{-1} - \gamma P_{22}\end{pmatrix} < 0.$$

That is

$$\begin{pmatrix} U^TQ_1^{-1}U - \gamma Q_1^{-1} & U^TQ_1^{-1}V & U^TQ_1^{-1}\begin{pmatrix} CE & -I\end{pmatrix}\\ V^TQ_1^{-1}U & V^TQ_1^{-1}V & V^TQ_1^{-1}\begin{pmatrix} CE & -I\end{pmatrix}\\ \begin{pmatrix}(CE)^T\\ -I\end{pmatrix}Q_1^{-1}U & \begin{pmatrix}(CE)^T\\ -I\end{pmatrix}Q_1^{-1}V & \begin{pmatrix}(CE)^T\\ -I\end{pmatrix}Q_1^{-1}\begin{pmatrix} CE & -I\end{pmatrix} - \gamma P_2\end{pmatrix} < 0.$$

Let $H = \begin{pmatrix} CE & -I \end{pmatrix}$, then

$$\begin{pmatrix} U^TQ_1^{-1}U - \gamma Q_1^{-1} & U^TQ_1^{-1}V & U^TQ_1^{-1}H \\ V^TQ_1^{-1}U & V^TQ_1^{-1}V & V^TQ_1^{-1}H \\ H^TQ_1^{-1}U & H^TQ_1^{-1}V & H^TQ_1^{-1}H - \gamma P_2 \end{pmatrix} < 0.$$

Pre- and postmultiplying above inequality by the symmetric matrix diag(Q_1, I, I) and its transpose, respectively, we have

$$\begin{pmatrix} Q_1U^TQ_1^{-1}UQ_1 - \gamma Q_1 & Q_1U^TQ_1^{-1}V & Q_1U^TQ_1^{-1}H \\ V^TQ_1^{-1}UQ_1 & V^TQ_1^{-1}V & V^TQ_1^{-1}H \\ H^TQ_1^{-1}UQ_1 & H^TQ_1^{-1}V & H^TQ_1^{-1}H - \gamma P_2 \end{pmatrix} < 0.$$

Obviously, it Is equivalent to the following inequality:

$$\begin{pmatrix} -\gamma Q_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\gamma P_2 \end{pmatrix}$$

$$- \begin{pmatrix} (UQ_1)^T \\ V^T \\ H^T \end{pmatrix} \left(-Q_1^{-1}\right) \begin{pmatrix} UQ_1 & V & H \end{pmatrix} < 0.$$

Since $-Q_1 < 0$, by lemma 1 we get

$$\begin{pmatrix} -\gamma Q_1 & 0 & 0 & Q_1U^T \\ 0 & 0 & 0 & V^T \\ 0 & 0 & -\gamma P_2 & H^T \\ UQ_1 & V & H & -Q_1 \end{pmatrix} < 0.$$

Replacing U and V by $I + CBF_e, CA + CBF_x$ in above inequality, respectively, one can follow that

$$\begin{pmatrix} -\gamma Q_1 & 0 & 0 & Q_1(I + CBF_e)^T \\ 0 & 0 & 0 & (CA + CBF_x)^T \\ 0 & 0 & -\gamma P_2 & H^T \\ (I + CBF_e)Q_1 & CA + CBF_x & H & -Q_1 \end{pmatrix} < 0.$$

$$\begin{pmatrix} -\gamma Q_1 & 0 & 0 & Q_1 + Q_1F_e^TB^TC^T \\ 0 & 0 & 0 & A^TC^T + F_x^TB^TC^T \\ 0 & 0 & -\gamma P_2 & H^T \\ Q_1 + CBF_eQ_1 & CA + CBF_x & H & -Q_1 \end{pmatrix} < 0. \quad (34)$$

Set $L = F_eQ_1$. By (34) one can see that (15) is finally converted to an equivalent LMI (29):

$$\begin{pmatrix} -\gamma Q_1 & 0 & 0 & Q_1 + L^TB^TC^T \\ 0 & 0 & 0 & A^TC^T + F_x^TB^TC^T \\ 0 & 0 & -\gamma P_2 & H^T \\ Q_1 + CBL & CA + CBF_x & H & -Q_1 \end{pmatrix} < 0.$$

Since $Q_1 = P_1^{-1}$, by (16) one can obtain

$$\frac{1}{\lambda_1}R^{-1} < Q_1 < R^{-1}. \quad (35)$$

So Q_1 needs to satisfy the condition

$$\frac{1}{\lambda_1}R^{-1} - Q_1 < 0.$$

Let $\lambda_1' = \frac{1}{\lambda_1}$, then (35) is converted into a computationally tractable condition (30).

Since $\lambda_1' = \frac{1}{\lambda_1}$, (18) can be written as

$$\frac{1}{\lambda_1'}\delta_e^2 + \lambda_2 \sum_{i=1}^{N} \lambda_0^i \delta_W^2 < \frac{\varepsilon^2}{\gamma^{N-1}}.$$

Since $\lambda_1' > 0$, by lemma 1, it follows that

$$\begin{pmatrix} \lambda_2 d^2 - \frac{\varepsilon^2}{\gamma^{N-1}} & \delta_e \\ \delta_e & -\lambda_1' \end{pmatrix} < 0.$$

This completes the proof.

V. Numerical Example

Consider the discrete-time system

$$\begin{aligned} x(k+1) &= Ax(k) + Ew(k) + Bu(k) \\ w(k+1) &= Sw(k) \\ y(k) &= Cx(k), \end{aligned}$$

where $A = \begin{pmatrix} 1 & 0 \\ 0.5 & 0.75 \end{pmatrix}, B = \begin{pmatrix} -0.2 & 0 \\ -0.1 & 1 \end{pmatrix}, E = \begin{pmatrix} 1 & 0.5 \\ -0.25 & 0.5 \end{pmatrix}, S = \begin{pmatrix} 0.6 & 0.3 \\ 0 & -0.9 \end{pmatrix}, C = \begin{pmatrix} 1 & -0.1 \\ 0.3 & 0.6 \end{pmatrix}$.

Let $\delta_e = 0.1, \delta_w = 0.1, \varepsilon = 11, R = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, N = 10$. The reference signal is $r(k) \in \mathbb{R}^q$ generated by the following system

$$r(k+1) = Mr(k) = \begin{pmatrix} 1 & 0 \\ 0 & 0.5 \end{pmatrix} r(k), \qquad r(0) = r_0,$$

and $\delta_r^2 = r^T(0)r(0) = 0.01$. Then by a simple calculation we have

$$\lambda_0 = \max\{\lambda_{max}(S^TS),\ \lambda_{max}(M^TM)\} = 1.$$

So $\delta_W^2 = 2(\lambda_0 + 1)\delta_w^2 + 2(\lambda_0^2 + \lambda_0 + 1)\delta_r^2 = 0.1$, and $d^2 = \sum_{i=1}^{N} \lambda_0^i \delta_W^2 = 1$.

Using the LMI toolbox in Matlab to solve the LMIs (29)-(32) in Theorem 2, the feedback gain matrices are given by

$$F_x = \begin{pmatrix} 5.0000 & 0.0000 \\ 0.0000 & -0.7500 \end{pmatrix},$$
$$F_e = LQ_1^{-1} = \begin{pmatrix} 4.7620 & 0.7939 \\ 0.9524 & -1.5080 \end{pmatrix}.$$

This numerical example illustrates the feasibility of the controller in Theorem 2.

REFERENCES

[1] [1] A. M. Lyapunov, "The general problem of the stability of motion," International Journal of Control, 1992, 55(3):531-534.

[2] [2] P. Dorato, Short time stability in linear time-varying systems, Proc. of IRE International Convention Record Part 4, 1961, pp. 83 C 87.

[3] [3]F. Amato, M. Ariola, C. T. Abdallah, et al, "Dynamic output feedback finite-time control of LTI systems subject to parametric uncertainties and disturbances," Control Conference. IEEE, 2015.

[4] [4]F. Amato, M. Ariola, C. T. Abdallah, et al, "Finite-time control for uncertain linear systems with disturbance inputs," American Control Conference. IEEE, 1999.

[5] [5] F. Amato, M. Carbone, M. Ariola, et al, "Finite-time stability of discrete-time systems," American Control Conference, 2004. Proceedings of the 2004. IEEE, 2004.

[6] [6]F. Amato, M. Ariola, M. Carbone, et al, "Control of linear discrete-time systems over a finite-time interval," IEEE Conference on Decision & Control. IEEE, 2005.

[7] [7]F. Amato, M. Ariola, C. Cosentino, "Finite-time control of discrete-time linear systems: Analysis and design conditions," Automatica, 2010, 46(5):919-924.

[8] [8]K. Fujimoto, T. Inoue, S. Maruyama, "On finite time optimal control for discrete-time linear systems with parameter variation," IEEE Conference on Decision & Control. IEEE, 2015.

[9] [9]W. Kang, S. Zhong, K. Shi, et al, "Finite-time stability for discrete-time system with time-varying delay and nonlinear perturbations," ISA Transactions, 2015, 60:67-73.

[10] [10]D. Rotondo, F. Nejjari, V. Puig, "Dilated LMI characterization for the robust finite time control of discrete-time uncertain linear systems," Automatica, 2016, 63:16-20.

[11] [11]K. Y. Lian, J. J. Liou, Output Tracking Control for Fuzzy Systems Via Output Feedback Design. IEEE Press, 2006.

[12] [12]M. Chen, S. S. Ge, B. Ren, Adaptive Tracking Control of Uncertain MIMO Nonlinear Systems with Input Constraints. Pergamon Press, Inc. 2011.

[13] [13]K. Park, J. H. Lee, S. H. Kim, et al, "Direct tracking control using time-optimal trajectories," Control Engineering Practice, 1996, 4(9):1231-1240.

[14] [14] A. P. Aguiar, J. P. Hespanha, Petar V. Kokotovi, "Performance limitations in reference tracking and path following for nonlinear systems," Automatica, 2008, 44(3):598-610.

[15] [15]S. Skogestad, I. Postlethwaite, Multivariable Feedback Control Analysis and Design, second ed., John Wiley & Sons, England, 2005.

[16] [16]E. Gershon, U. Shaked, I. Yaesh, 4 Continuous-time Systems: Tracking Control, H_∞ -Control and Estimation of State-multiplicative Linear Systems. Springer London, 2005.

[17] [17]K. Ogata, Discrete Time Control Systems, Second Edition, Control Engineering Series, Marcel. Facta universitatis - series: Electronics and Energetics, 2010, 84(1):11-20.

[18] [18]F. Liao, Y. Wu, et al, "Finite-time bounded tracking control for linear discrete-time systems," Mathematical Problems in Engineering, Volume 2018, Article ID 7017135.