Detection of Pseudo CRS Frontier for a Negative Data Using RTS Model of Allahyar & Modified Multiplier BCC Model

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DETECTION OF PSEUDO CRS FRONTIER FOR A NEGATIVE DATA USING RTS MODEL OF ALLAHYAR & MODIFIED MULTIPLIER BCC MODEL

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ABSTRACT:

Performance measurement of Decision Making Units (DMU) possessing an array of positive and negative type of data has been an extensively researched topic in Data Envelopment Analysis. However, assessment of Returns to Scale (RTS) under negative data problem is rarely witnessed without the steps referred by Allahyar, M. (2015). Authors purported a solution around the vicinity of the Decision Making Unit under examination to predict the nature of the Return to Scale of a firm. The extant investigation is aimed to extend the research of Allahyar, M. (2015) to identify a Pseudo Frontier for a negative data problem under Constant Return to Scale. In addition to it, a new origin based on the provided data is also computed with a view to convert the entire data set into a positive dataset. However, this approach seems to be ineffective to create a frontier under the multiple input output scenario. In this regard, a new variation of the Multiplier form of BCC model is proposed here to detect the new origin for the sake of designing the Pseudo CRS Frontier. Small examples are added for the elaboration of the CRS efficient DMUs using methods described by Allahyar, M. (2015) and identification of the New Origin from the Multiplier form of BCC model.

Keywords: *Data Envelopment Analysis, Negative Data, Constant Return to Scale, Pseudo Frontier*

1. INTRODUCTION:

Returns To Scale has been a useful domain of research in the field of Micro-Economics to manifest the effects of long-term changes in the factor of production on the output set. The tenet was found inevitable for assessing long-term average cost of a firm. In a nutshell, the prime focus of the study was whether factors were to be scaled up to achieve Economy of

Scale due to Increasing Returns To Scale or not so that long term average cost can be lowered further. In this regard, researchers conceived homogenous functions and transformation functions so that the effects on the output set could be tracked by estimating "Elasticity of Scale". These efforts acknowledged the presence of premeditated theoretical production frontier. However, techniques of measurement were required to be modified to reconcile with an empirical curve designated by Data Envelopment Analysis (DEA).

The journey of DEA commenced when the performance of students from participating and not participating schools were compared by Charnes et al. (1978) by means of a data-oriented, linear programming-based, nonparametric approach described earlier by Farrell M. J. (1957). However, until the year of 1989, major theories alluded by Charnes et al. (1978), Banker et al. (1984), Brockett et al (1997) etc., in the arena of DEA**,** postulated the use of nonnegative data. Pastor (1993) was the first who took the initiative for solving a negative data problem. He applied the theorems of “Translation Invariance” (originated by Ali and Seiford (1990) (and Cooper et al (2011)) for measuring the performances of 23 bank branches. Moreover, he showed (Pastor (1996)) that a Relocation of Origin, to another neighbouring point, does not alter the efficient frontier translation invariant DEA formulations. Thrall R. M. (1996)), however, highlighted the impact of the translation invariant forms on the “optimal dual solutions”. Halme et al., (1998) recommended an interval variable with the subtraction of two ratio scale variables. An extra output (input) was created for each negative input scale (output) variable. Scheel (2001) (Emrouznejad, A., Anouze, A. L. (2010)), later on, gave a similar treatment to non-positive input (output). An overview of the negative data problem can be seen in Pastor J. T., Ruiz, J. L., (2007) as mentioned by Zhu, J. & Cook, W. D. in their book.

Although, the model could derive the radial efficiency scores but it failed to reflect the impact in terms of true Production Frontier. In this context, Emrouznejad A., Anouze A. L. (2010) proposed a unique way to measure the semi-oriented radial efficiency score. Any variable with mixed data was expressed in terms of the subtraction of two variables having nonnegative data. However, it could not ensure the Pareto Efficient Solution, but was able to handle a negative part of a variable in a positive format. Matin R. K, Azizi R. (2010) presented a new two-phase approach based on a modified version of the classical additive DEA model, which was designed to provide target with nonnegative value for each observed DMU.

Chambers, R.G., Chung, Y., Fare, R., (1996) explored a relationship among the distance function proposed by Shephard (1953) and the Benefit function introduced by Luenberger D. G. (1992). These authors were able to express the later one as a directional distance input function for the characterization of a technology in terms of price and input space. In view of handling negative data, the directional distance model, prescribed by Chambers, R.G., Chung, Y., Fare, R., (1998), was applied by Portela et al. (2004) for measuring performances of all branches of a Portuguese bank. This highly acclaimed Range Directional Model (RDM) is a unique variation of Relocation Policy. The efficiency of a firm is measured in comparison to the deviation seen from a pre-defined Ideal point (Superior Origin). The research of Cheng, G. et al (2011) can be included under the category of Directional Distance Function which was able to achieve similar outputs as obtained from other radial models. The direction vector was kept proportional to the absolute value of the input-output vector of DMU.

Sharp et al. (2006) introduced a modified slack-based model (MSBM) which was both unit invariant and translation invariant in nature. Negative data problem also had a unique problem of detecting the Return to Scale (RTS) for a Decision Making Unit under observation. Return To Scale is measurement of increase in output bunch due to an unit increase in the input bunch. This would mean that if the input bunch is doubled then for a constant return to scale the output set also gets increased by two times. However, doubling a negative input along with another positive input do not mean in the same manner. VRS approach has always been widely accepted and acknowledged by researchers as a CRS would lead to contraction or expansion of the activity of any firm (Fare et al (1994)). Portela et al (2004) cited an example to display the inherent fallacy of the model. Allahyar, M. and Malkhalifer, M. R., (2015) presented an approach to resolve the issue of RTS. They induced the theme of a neighbourhood analysis to observe the changes in the output and input sets.

The brief of this contemporary research hovers around two points. One group (such as Pastor (1996) and Portela et al (2004)) favoured the relocation of origin and permits the transformation of the data set to allow the application of regular methods. Some even stayed with the same origin and proposed models to obtain VRS efficient DMUs. However, no one even patronized the possibility of a Pseudo frontier owing to CRS. The extant tenet is clarifying its existence and prescribes the true origin for the given data. The entire study is therefore an extension of the model of Allahyar, M. and Malkhalifer, M. R., (2015) for designing a Pseudo Frontier in congruence with the CRS policy. In brief, the effort is made to obtain a new origin to allow the factors of a firm to be scaled up in a proportional manner.

2. PROBLEM STATEMENT:

The application of CRS on the presence of negative data has been denied by many authors. Portela et al (2004) typically insinuates the reason of failure with a small example. According to the authors, in such instances radial directions are unable to predict an appropriate peer which had to remain superior in both spheres of input utilization and production of outputs.

The entire episode led to two major questions (in the context of negative data):

- Allahyar, M. and Malkhalifer, M. R., (2015) accepted a small change in the input or output vector to confirm the Return to Scale of a firm. It also simultaneously affirms the existence of CRS efficient DMUs. However, will this theory aid in building the foundation of a Pseudo CRS frontier?
- Secondly, is it possible to obtain an unambiguous origin which would be the initiating point for the Pseudo CRS Frontier?

3. DERIVATION OF PSEUDO CRS MODEL FROM ALLAHYAR'S MODEL:

3.1. DETERMINATION OF SLOPE OF THE PSEUDO CRS FRONTIER:

Given a homogenous production function, CRS frontier is oriented on the basis of the subsequent equality are allowed strictly positive values for both input (x) & output (y) (shown below):

$$\frac{dx}{x} = \frac{dy}{y} = \omega$$

This equation iterates the necessary & sufficient condition of having a CRS technology in existence. As the rate of % change (rise or fall) of the input are same as % change of the output. The solution of this differential equation leads to a linear equation $y = mx$ (where m is the slope of the straight line) which moves across the origin. The solution itself characterises the production frontier. However, the scenario gets changed if any one of them is converted to negative. Though, the resulting solution, $y = m|x| \; for \; m, y > 0$, passes through the origin, but would reflect a connectivity among a desirable output and an undesirable input. Hence, for a preconceived production frontier (which is concave in nature) the slope at a certain point is expected to satisfy the condition mentioned below:

$$\left[\frac{dy}{dx}\right]_{x=x_0} = \left[\frac{y}{|x|}\right]_{x=x_0} \; for \; x < 0 \,\&\, y > 0$$

The model suggested by Allahyar, M. (2015) perhaps was aligned as per following rule to eradicate this crisis owing to a negative data:

$$\frac{dx}{|x|} = \frac{dy}{|y|} = \omega \text{ or } \frac{dy}{dx} = \frac{|y|}{|x|}$$

It literally accepts the philosophy of increasing or decreasing the negative input or output by adding or subtracting an amount which proportional to the absolute value of the corresponding input or output. The tenets of Allahyar, M. (2015) for detecting the return to scale for a negative data problem to deal with the investigation around the strongly efficient VRS based DMUs while employing the subsequent models.

<table>
<tr><td>Right side of Production Frontier</td><td>Left side of Production Frontier</td></tr>
<tr><td>$Max\ Z = \theta_0$
$\sum_{i=1}^{c} \lambda_i x_{ij} \leq x_{oj} + \delta|x_{oj}|,\ \forall j$
$\sum_{i=1}^{c} \lambda_i y_{ik} \geq y_{ok} + \theta_0|y_{ok}|,\ \forall k$
$\sum_{k=1}^{c} \lambda_k = 1$
$y_{ik} \in R^{\pm}, x_{ij} \in R^{\pm}, \theta_0 \in R^{\pm}$ ... (1A)</td><td>$Max\ Z = \theta_0$
$\sum_{i=1}^{c} \lambda_i x_{ij} \leq x_{oj} - \theta_0|x_{oj}|,\ \forall j$
$\sum_{i=1}^{c} \lambda_i y_{ik} \geq y_{ok} - \delta|y_{ok}|,\ \forall k$
$\sum_{k=1}^{c} \lambda_k = 1$
$y_{ik} \in R^{\pm}, x_{ij} \in R^{\pm}, \theta_0 \in R^{\pm}$ ... (1B)</td></tr>
</table>

Based on the optimal scores of θ_0 *and* the predefined small value of δ the status of a DMU is decided. The modus operandi of the model (for a single negative input and single output problem) is briefly explained in Figure 1a & 1b.

A firm would be categorised as Efficient under Constant Return to Scale (CRS) when any of the following cases occur:

- if the DMU follows IRS on its left side and subsequently adopts DRS on the Right hand side. Here, IRS implies that if the input is increased by a certain percentage then the output will increase at a higher percentage. On the contrary, DRS is realised if the output is reduced by a certain rate then the input will decrease at a lower rate (mentioned in Figure 1a).
- if anyone side of the DMU follows CRS (when the rate of increase or decrease will transpire equally for both input & output (mentioned in Figure 1b)).

<Insert Figure 1a, 1b>

The second condition (Figure 1b) provides a clear picture of the origin as the entire facet depicts the linear equation of CRS frontier. Extension of this facet can point out the location of the new origin. However, the first case does not offer any clear message about the location of the origin. It could be within any two intersecting points obtained from the extension of the lines AB and AC with input axis. Under such circumstances, the new origin is obtained using the principle of Allahyar, M. (2015). A CRS efficient DMU can only be created outside the boundary spanned by VRS by increasing input and output at an equal rate. The line joining the current DMU and the new point will have a positive slope and will certainly intersect the abscissa to create a new origin.

To locate the frontier (for a single input and single output problem) and to reveal the new origin models depicted in 1a and 1b are reoriented. As per the sign convention, a VRS efficient DMU-O can have four input output combinations such as ($x_{ij}, y_{ik} \in R^{+}$), ($x_{ij}, y_{ik} \in R^{-}$), ($x_{ij} \in R^{-}, y_{ik} \in R^{+}$) and ($x_{ij} \in R^{+}, y_{ik} \in R^{-}$). The third type (which is ($x_{ij} \in R^{-}, y_{ik} \in R^{+}$)) of the sign restriction will lead to two subsequent forms:

Right side of Production Frontier	Left side of Production Frontier
$Max\ Z = \theta_0$ $\sum_{i=1}^{c} \lambda_i{}' x_{ij} \le x_{oj},\ j = 1$ $\sum_{i=1}^{c} \lambda_i{}' y_{ik} \ge y_{ok}(1 + \theta_0 + \delta),\ k = 1$ $\sum_{k=1}^{c} \lambda_i{}' = 1 + \delta$ $\lambda_i{}' = \begin{Bmatrix} \lambda_o + \delta & i = o \\ \lambda_i & i \ne o \end{Bmatrix}$ $y_{ik} \in R^{+}, x_{ij} \in R^{-}, \theta_0 \in R^{\pm}$ ... (2A)	$Max\ Z = \theta_0$ $\sum_{i=1}^{c} \lambda_i{}' x_{ij} \le x_{oj}(1 + \theta_0 + \delta),\ j = 1$ $\sum_{i=1}^{c} \lambda_i{}' y_{ik} \ge y_{ok},\ k = 1$ $\sum_{i=1}^{c} \lambda_i{}' = 1 + \delta$ $\lambda_i{}' = \begin{Bmatrix} \lambda_o + \delta & i = o \\ \lambda_i & i \ne o \end{Bmatrix}$ $y_{ik} \in R^{+}, x_{ij} \in R^{-}, \theta_0 \in R^{\pm}$ ... (2B)

The choice of the positive value of δ (in model 2a and 2b) will make the transition of both problems from VRS towards a DRS. Now, if a single CRS efficient DMU is found in such a single input & single output problem (mentioned in Figure 1a) then following derivations are needed to identify the slope of the Pseudo Frontier.

Let, two optimal solutions obtained from the above two models are referred as $\theta_{0i}{}^{*}(\delta)$ *and* $\theta_{Oo}{}^{*}(\delta)$. Eventually, it will result a pair of peers denoted as follows:

$$\left(x_o, y_o(1 + \theta_{Oo}{}^{*}(\delta) + \delta)\right) \text{ and } \left(x_o(1 + \theta_{0i}{}^{*}(\delta) + \delta), y_o\right)$$

These two points will certainly be useful to detect the slope of the CRS. Hence, the slope of the CRS frontier will be computed as follows:

$$S = \frac{\left(y_o\left(1+\theta_{Oo}{}^*(\delta)+\delta\right)-y_o\right)}{\left(x_o - x_o\left(1+\theta_{Oi}{}^*(\delta)+\delta\right)\right)} = \left(\frac{y_o}{|x_o|}\right)\left(\frac{\theta_{Oo}{}^*(\delta)+\delta}{\theta_{Oi}{}^*(\delta)+\delta}\right)$$

However, according to Allahyar, M. (2015) any CRS efficient DMU can only remain efficient if the equality of $\theta_{Oo}{}^*(\delta) = \theta_{Oi}{}^*(\delta) = \delta$ exists. Hence, it can be ensured that any CRS efficient DMU would possess a slope equivalent to $S = \left(\frac{y_o}{|x_o|}\right)$.

Similarly, forth condition will present a slope $S = \left(\frac{|y_o|}{x_o}\right)$ along with a new origin located at $(0,2y_o)$. On the other hand, the first and second conditions give rise to slopes of $S = \left(\frac{y_o}{x_o}\right)$ $and\ S = \left(\frac{|y_o|}{|x_o|}\right)$ while having the origin situated at (0, 0).

Considering (x_o, y_o) as a CRS efficient DMU and the line drawn thorough it at a slope of S will certainly locate the new origin on the x axis. Relocating the present origin to this new point will turn the negative data into a positive data. In this case (for single negative input and single nonnegative output), the coordinate of this new origin will be $(2x_o, 0)$. The following theorem is referred to ensure this proposition:

Theorem 1: A single negative input & positive output scenario can only suffice the existence of a single CRS efficient Decision Making Unit

Proof: Let there are at least two CRS efficient DMUs (say, t^{th} and u^{th}) situated on the production frontier. As a result, the slopes obtained owing to these points will be $S_t = \left(\frac{y_t}{|x_t|}\right)$ $and\ S_u = \left(\frac{y_u}{|x_u|}\right)$. Moreover, these slopes will be equivalent to each other $(S_t = S_u)$.

$$\left(\frac{y_t}{|x_t|}\right) = \left(\frac{y_u}{|x_u|}\right)$$

Since, both units are different from each other, so, it is assumed that the input output vectors are not identical to each other. In this regard, two inequalities such as $x_t < x_s$ and $y_t < y_s$ are considered here. But, it these relationships also approve that $|x_t| > |x_s|$ and $y_t < y_s$ which consequently, affirms an inequality proposition shown as $\left(\frac{y_t}{|x_t|} < \frac{y_s}{|x_s|}\right)$. Hence, there can only be one and only one CRS efficient DMU under this condition. Hence, it fulfils the scenario stated in Figure 1a.

However, the second scenario (mentioned in Figure 1b (when any one side follows CRS)) does require this condition to be fulfilled by any one point located on the facet. On the other hand, the other point on the facet will automatically be located on the frontier when it is drawn from the new origin.

3.2 PROBLEM WITH THE METHOD FOR DETECTING CRS EFFICIENT DMU

Though, the extended model of Allahyar, M. (2015) is justified for any type of single input single output data, however, it seems to be unproductive to resolve problems which are pertaining to the multi-input output scenario. Subsequent points have been observed which prevented the application of this model under such multiple input output problems:

- In a multi-input output negative data problem there can be a number of CRS efficient members. Each of these members will then have their individual input and output directions. Hence, it is very hard to state which one of them is to be chosen. It will certainly prevent the selection of the justified slope which would be utilised to explore the new origin.
- These directions do not guarantee of converging to a single point.

In this regard, a new variation of BCC multiplier model is proposed to serve the purpose. The imminent question may arise from here:

- Is it possible to establish these facts using envelopment or a multiplier model?

The sole reason for choosing a Modified Multiplier BCC Model is owing to the unique characteristics of it. For any non-negative data set the model has an immense capability to distinguish CRS efficient DMUs from other VRS efficient members. Moreover, the CRS frontier drawn from any CRS efficient must pass through the origin. Hence, if a number of CRS efficient firms are detected from the analysis of Allahyar, M. (2015) then the proposed model would produce the same number of planes passing through the common point (which is termed here as new origin).

4. MULTIPLIER MODELS FOR NEGATIVE DATA:

This segment a generalised multiplier model is demonstrated to allow any type of multi-input & output data. Bearing in mind to the previously stated proofs and facts, five combinations such as strictly positive inputs and outputs, strictly negative inputs and outputs, strictly positive inputs and mixed outputs and mixed inputs and strictly positive outputs, are created. Each of these varieties is needed to be furnished with their own specific models.

Case 1: Multiplier Model for strictly positive inputs and outputs

The linear equation, $ax = cy$ is used to create the production frontier for a single input (x) output (y) problem. Pursuing the same principle the model will give rise to original Multiplier Model of DEA originated by CCR.

Case 2: Multiplier Model for strictly negative inputs and outputs

In this case, the production frontier is defined by the linear equation, $ax + b_1 = cy + b_2$, where $a, c, b_1, b_2 > 0$. The original Multiplier Model of DEA originated by CCR will be enough for determining efficiency scores and Pseudo CRS frontier.

Case 3: Multiplier Model for strictly positive inputs and mixed outputs

Conceiving the single input output scenario, the frontier will certainly change the nature of its intercept. The frontier will pursue a linear equation demonstrated as $ax = cy + b$, where $a, c > 0$ *and* b to be an unrestricted parameter. Hence, the final model in this case is given below:

$Max\ Z = \sum_{i=1}^{n_o} c_i y_{io} + b$
$\sum_{i=1}^{n_o} c_i y_{ij} + b \le \sum_{i=1}^{n_i} a_i x_{ij}$
$\sum_{i=1}^{n_i} a_i x_{io} = 1$
$\sum_{i=1}^{n_o} c_i y_{ij} + b \ge 0$
$y_{ij} \in R^+,\ x_{ik} \in R^-$
$a_i, c_i, b \ge 0,\ i = 1,2, \dots n_D, ,\ k = 1,2 \dots n_o, j = 1,2 \dots n_i$ ...(3A)

Case 4: Multiplier Model for mixed inputs and strictly positive outputs

In this scenario, model (3B) mentioned above will be very effective.

$Max\ Z = \sum_{i=1}^{n_o} c_i y_{io}$
$\sum_{i=1}^{n_o} c_i y_{ij} \le \sum_{i=1}^{n_i} a_i x_{ij} + b$
$\sum_{i=1}^{n_i} a_i x_{io} + b = 1$ such that $b \ge 1 + \Delta$ & $\Delta \approx 0$
$\sum_{i=1}^{n_i} a_i x_{ij} + b \ge 0$
$y_{ij} \in R^+,\ x_{ik} \in R^-$
$a_i, c_i, b \ge 0,\ i = 1,2, \dots n_D, ,\ k = 1,2 \dots n_o, j = 1,2 \dots n_i$ ...(3B)

Case 5: Multiplier Model for mixed inputs and outputs

In case of a mixed multi-input output problem, a CRS efficient DMU may possess any type of input output vector. In this regard, two nonnegative variables say, b_1 *and* b_2 are addressed in the model. b_1 may be equivalent to b_2 or may not be. These variables will be instrumental for locating the new origin for the given problem.

$Max\ Z = \sum_{i=1}^{n_o} c_i y_{io} + b_2$
$\sum_{i=1}^{n_o} c_i y_{ij} + b_2 \leq \sum_{i=1}^{n_i} a_i x_{ij} + b_1,$ $\sum_{i=1}^{n_o} c_i y_{ij} + b_2 \geq 0$ $\sum_{i=1}^{n_i} a_i x_{ij} + b_1 \geq 0$ $\sum_{i=1}^{n_i} a_i x_{io} + b_1 = 1$ such that $b_1 \geq 1 + \Delta,\ for\ \Delta \approx 0$ $y_{ij} \in R^{\pm},\ x_{ik} \in R^{\pm}$ $a_i, c_i, b_1, b_2 \geq 0, i = 1,2, \ldots n_D, ,\ k = 1,2 \ldots n_o, j = 1,2 \ldots n_i$ ...(3C)

However, it has to be kept in mind that for the sake of declaring a DMU as CRS efficient one has to utilise the method proposed by Allahyar, M. (2015). The search of the origin will be followed once the detection is over. In addition to it, in each model stated above should exclude positive inputs and outputs from the nonnegative constraints. This would ensure the constants for the negative or mixed type of variables.

4.1 SELECTION OF THE NEW ORIGIN:

The selection process is meant to choose one from a large pool of points (as there can be large number of points which would be considered as legitimate points). The key objective of including such a process is to ensure that the slopes of the lines joining the new origin and the CRS efficient DMUs should be more than any other VRS efficient DMU. The above mentioned variation of a regular BCC based multiplier model is aimed to select one of them with a few constraints of non-negativity for handling negative data.

- If single CRS efficient DMU is obtained from a negative data problem then the intercept on the input & output axes are determined from the parametric values of a_i, c_i *and* b derived from the model. The point of intercept on the input axis will be located at $(-\emptyset_k, 0)$ $when\ \emptyset_k > 0$. The value of $\emptyset_k$ can be determined by the ratio of the dual values $\frac{b}{a_k}$.

- For a large number of CRS efficient DMUs, the equations obtained from the modified BCC model are employed to determine the new origin. The input side of the linear equation due to each CRS efficient DMU is set as zero. The solution from these independent equations will lead to the location of the new origin for the input side. Similar steps are prescribed for the output side as well.

5. EXAMPLE:

5.1 Case with Single Negative Input & Single Positive Output:

To realise the effectiveness of the model an example of single negative input data along with a positive output is brought under consideration (Table 1). The graphical view of this example (mentioned in Figure 2) acknowledges the presence of two VRS efficient firms (B (-15, 39) and C (-27, 27)).

<Insert Table 1: Data Table>

<Insert Figure 2: Example>

Application of the model prescribed by Allahyar, M. (2015) confirms the presence of CRS for the firm C only. Objective scores indicate the presence of IRS and CRS on its left and right side. But, most importantly, B displays the existence of DRS on its both sides.

<Insert Table 2: Detection of RTS for the VRS Efficient DMUs >

Pursuing this conclusion the slope as well as the point of interception on x axis are found to be S = 1 and (-54, 0) respectively. Relocating the origin from (0, 0) to (-54, 0) results a new dataset cited in Table 3. A mere ratio test is conducted on this data to compute the efficiency scores of each of these DMUs. It can be observed that though the formal approach can reveal only one CRS efficient DMU but present numeric values of B have also allowed it to become another member of the same group.

<Insert Table 3: CRS Efficiency Score of DMUs >

On the contrary, a strange phenomenon is observed if the outcome of the prescribed multiplier model on the VRS efficient DMUs (B and C) is scrutinised. C is the only one member which is able score all positive coefficients for the linear equation $L_1x + L_3 = L_2y$ used for defining the Pseudo CRS production frontier. Here, (x, y) is the input output vector of the contesting members. Most importantly, the intercept obtained from the optimal solution of C given in Table 4 is $\frac{L_3}{L_2} = 54$ (which is same as the previous value obtained from the envelopment model). Hence, the multiplier model is found quite reliable to identify the CRS frontier and CRS efficient DMUs. The success of this multiplier model is solely attributed to the value of L_3 which remains above one in both cases. Hence, in a nutshell, the problem involving mixed input data should have a constraint $L_3 \geq 1 + \Delta$ where $\Delta \approx 0$.

<Insert Table 4: Outputs of Pseudo CRS Multiplier Model>

The proposed multiplier model in this case will raise a question whether the entire theory can be pertinent to a mixed type of Multi-Input & Output problem or not.

5.2 Case with Mixed type of Multi-Input & Output Problem:

The proposed model (3C) is required to be applied on the VRS efficient DMUs obtained from the SBM. C, E, G, H & J are found to be efficient among the 10 DMUs (Table 5) depicted in the Table 6.

<Insert Table 5: Data Table 2>

<Insert Table 6A, 6B: SBM Output>

The consequence of the application of (3C) (shown in Table 7) portrays two varieties of linear equations (Table 8). These two emerged from E & H whereas the remaining DMUs did have zero values as the optimal parametric values. In other words, these parametric values do provide required information about the Pseudo CRS frontier.

<Insert Table 7: Optimal Table>

<Insert Table 8: Coefficients of the Model Parameters>

Table 8 primarily reflects the information about the new origin. Such as in both cases L5 has a same value of 18. Hence, *y* has to initiate from -18. Similarly, the coordinates belong to two inputs are obtained after solving two equations shown below:

$$0.3765x_1 + 0.2258x_2 + 53.003 = 0$$

$$0.0503x_1 + 0.2634x_2 + 46.046 = 0$$

Solution of these two equations is equivalent to $x_1 = 40.714$ *and* $x_2 = 166.857$. Hence, the new origin will be $(-40.714, -166.857, -18)$. The data mentioned in Table 2 is needed to be transformed to create a new positive data set which can endure through any CRS model

(Table 9). The last column of Table 9 confirms that the slopes pertaining to the CRS efficient DMUs were larger than the remaining VRS efficient DMUs. Finally, the outcome of the Input oriented CRS model on this data is as per the notion depicted before. E & H are the only candidates which could become CRS efficient DMUs. Even though attaining a VRS efficient status, C, G and J stayed as CRS inefficient members.

<Insert Table 9: Transformed Data>

<Insert Table 10: Outcome of IO-CRS Model>

6. CONCLUSION:

The entire study hovers around two points. Firstly, it resolves that the DMU displaying the CRS is truly a Pseudo CRS Efficient member. In other words, it is situated on the CRS frontier drawn from the new origin defined by the model. Secondly, it proves that the true CRS can only exist when the input output vector remains either positive or negative. But, in other two cases the CRS frontier is to be initiated from a point other than the real origin. Hence, it can be stated that CRS frontier always exists but depending on the nature of the data the origin needs to be shifted. At the end, the ramification of searching for the new origin can bring forth several benefits such as the computation of scale efficiency, identification of the direction along which scaling can be accomplished, ease of analysing productivity when dealing with the negative data.

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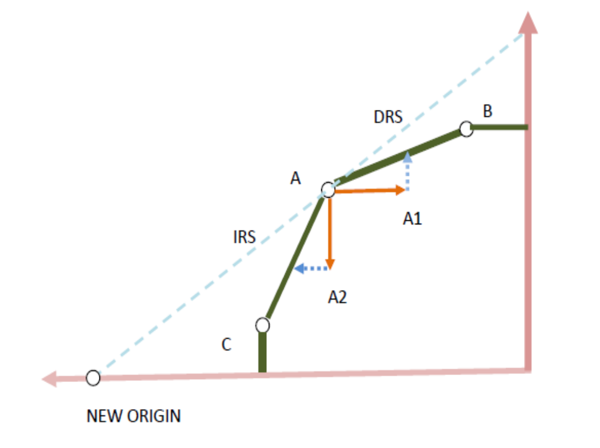

Figure 1A: CRS due to IRS & DRS

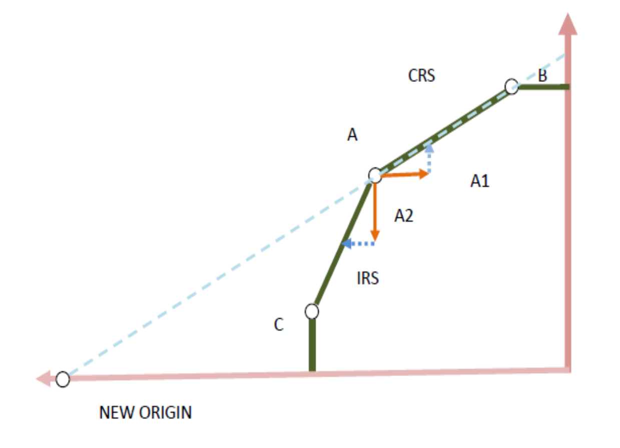

Figure 1B: CRS due to IRS & CRS

Table 1: Data Table

	INPUT	OUTPUT
A	-10	20
B	-15	39
C	-27	27
D	-20	20
E	-18	28
F	-17	23

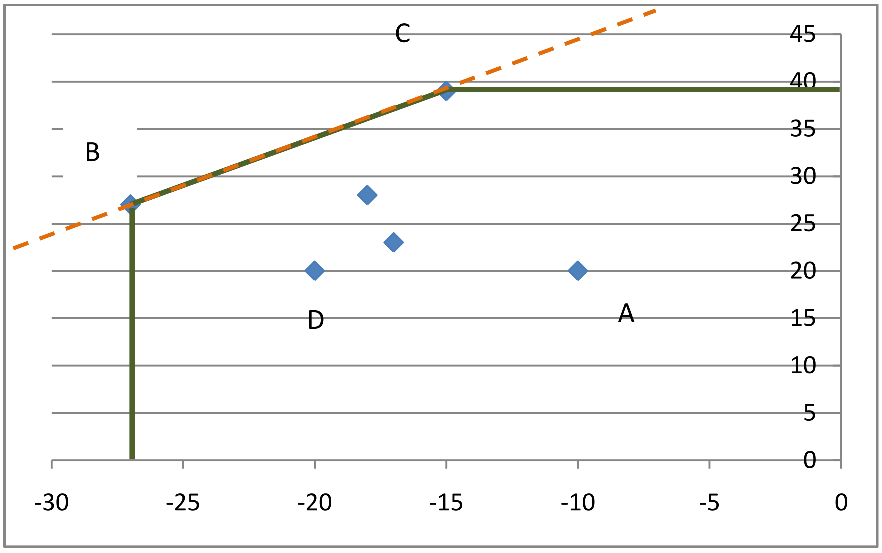

Figure 2: Example

Table 2: Detection of RTS for the VRS Efficient DMUs

DMU C		RHS	DMU B	LHS	DMU C	RHS	DMU B	LHS
Variable	Value	Reduced Cost	Value	Reduced Cost	Value	Reduced Cost	Value	Reduced Cost
A	0.001	0.000	0.000	0.000	0.000	0.000	0.003	0.000
L1	0.000	0.889	0.000	0.487	0.000	0.630	0.000	1.600
L2	0.002	0.000	1.000	0.000	0.000	0.444	0.997	0.000
L3	0.998	0.000	0.000	0.308	1.000	0.000	0.003	0.000
L4	0.000	0.519	0.000	0.487	0.000	0.259	0.000	0.933
L5	0.000	0.296	0.000	0.282	0.000	0.333	0.000	0.533
L6	0.000	0.519	0.000	0.410	0.000	0.370	0.000	0.933
C	0.001	0.000	0.001	0.000	0.001	0.000	0.001	0.000
Row	Slack or Surplus	Dual Price	Slack or Surplus	Dual Price	Slack or Surplus	Dual Price	Slack or Surplus	Dual Price
1.000	0.000	0.037	0.015	0.000	0.000	0.037	0.000	0.067
2.000	0.000	-0.037	0.000	-0.026	0.027	0.000	0.000	-0.067
3.000	0.000	2.000	0.000	1.000	0.000	1.000	0.000	3.600
4.000	0.000	1.000	0.000	0.000	0.000	0.000	0.000	2.600
	CRS		IRS		DRS		DRS	

Table 3: CRS Efficiency Score of DMUs

DMU	INPUT	OUTPUT	EFFICIENCY
A	44	20	0.455
B	39	39	1.000
C	27	27	1.000
D	34	20	0.588
E	36	28	0.778
F	37	23	0.622

Table 4: Outputs of Pseudo CRS Multiplier Model

	DMU B		DMU C	
Variable	Value	Reduced Cost	Value	Reduced Cost
Z	1.000	0.000	1.000	0.000
L2	0.026	0.000	0.037	0.000
L3	1.015	0.000	2.000	0.000
L1	0.001	0.000	0.037	0.000
Row	Slack or Surplus	Dual Price	Slack or Surplus	Dual Price
1.000	0.000	1.000	0.000	1.000
2.000	0.492	0.000	0.889	0.000
3.000	0.000	-1.000	0.000	0.000
4.000	0.296	0.000	0.000	-1.000
5.000	0.482	0.000	0.519	0.000
6.000	0.279	0.000	0.296	0.000
7.000	0.408	0.000	0.519	0.000
8.000	0.000	1.000	0.000	1.000
9.000	0.000	0.000	0.036	0.000
10.000	0.025	0.000	0.036	0.000

Table 5: Data Table 2

	INPUT 1	INPUT 2	OUTPUT
A	-10	4	21
B	-15	10	-17
C	-27	-5	-6
D	-20	12	18
E	-18	30	35
F	-17	-2	-18
G	-12	-1	26
H	-21	4	28
I	-19	6	-11
J	-16	-5	10

Table 6A: Outputs of SBM

	A		B		C		D		E	
Variable	Value	R. Cost	Value	R. Cost	Value	R. Cost	Value	R. Cost	Value	R. Cost
Z	18.000	0.000	57.000	0.000	0.000	0.000	19.000	0.000	0.000	0.000
S1	11.000	0.000	6.000	0.000	0.000	0.455	1.000	0.000	0.000	0.000
S2	0.000	1.200	6.000	0.000	0.000	1.808	8.000	0.000	0.000	0.000
S3	7.000	0.000	45.000	0.000	0.000	0.000	10.000	0.000	0.000	3.143
L1	0.000	18.000	0.000	18.000	0.000	23.000	0.000	18.000	0.000	40.000
L2	0.000	64.200	0.000	57.000	0.000	70.576	0.000	57.000	0.000	198.429
L3	0.000	8.200	0.000	19.000	1.000	0.000	0.000	19.000	0.000	125.857
L4	0.000	28.600	0.000	19.000	0.000	33.919	0.000	19.000	0.000	50.429
L5	0.000	53.200	0.000	22.000	0.000	70.374	0.000	22.000	1.000	0.000
L6	0.000	36.800	0.000	44.000	0.000	34.970	0.000	44.000	0.000	188.571
L7	0.000	0.000	0.000	6.000	0.000	1.051	0.000	6.000	0.000	12.286
L8	1.000	0.000	1.000	0.000	0.000	0.000	1.000	0.000	0.000	0.000
L9	0.000	45.400	0.000	43.000	0.000	47.525	0.000	43.000	0.000	165.571
L10	0.000	3.200	0.000	14.000	0.000	0.000	0.000	14.000	0.000	70.571
Row	Slack/ Surplus	Dp	Slack/ Surplus	D P	Slack/ Surplus	D. P	Slack/ Surplus	D. P	Slack/ Surplus	D. Price
1.000	0.000	1.000	0.000	1.000	0.000	1.000	0.000	1.000	0.000	1.000
2.000	0.000	1.000	0.000	1.000	0.000	1.455	0.000	1.000	0.000	1.000
3.000	0.000	2.200	0.000	1.000	0.000	2.808	0.000	1.000	0.000	1.000
4.000	0.000	-1.000	0.000	-1.000	0.000	-1.000	0.000	-1.000	0.000	-4.143
5.000	0.000	40.200	0.000	45.000	0.000	47.313	0.000	45.000	0.000	133.000

Table 6B: Outputs of SBM

	F		G		H		I		J	
Variable	Value	R. Cost	Value	R. Cost	Value	R. Cost	Value	R. Cost	Value	R. Cost
Z	34.857	0.000	0.000	0.000	0.000	0.000	43.000	0.000	0.000	0.000
S1	0.000	0.286	0.000	0.286	0.000	0.455	2.000	0.000	0.000	0.455
S2	0.000	1.714	0.000	1.714	0.000	1.808	2.000	0.000	0.000	1.808
S3	34.857	0.000	0.000	0.000	0.000	0.000	39.000	0.000	0.000	0.000
L1	0.000	21.143	0.000	21.143	0.000	23.000	0.000	18.000	0.000	23.000
L2	0.000	69.000	0.000	69.000	0.000	70.576	0.000	57.000	0.000	70.576
L3	0.000	1.857	0.000	1.857	0.000	0.000	0.000	19.000	0.000	0.000
L4	0.000	33.000	0.000	33.000	0.000	33.919	0.000	19.000	0.000	33.919
L5	0.000	67.429	0.000	67.429	0.000	70.374	0.000	22.000	0.000	70.374
L6	0.000	34.857	0.000	34.857	0.000	34.970	0.000	44.000	0.000	34.970
L7	0.107	0.000	1.000	0.000	0.000	1.051	0.000	6.000	0.000	1.051
L8	0.286	0.000	0.000	0.000	1.000	0.000	1.000	0.000	0.000	0.000
L9	0.000	47.000	0.000	47.000	0.000	47.525	0.000	43.000	0.000	47.525
L10	0.607	0.000	0.000	0.000	0.000	0.000	0.000	14.000	1.000	0.000
Row	Slack/ Sur	D. P	Slack/ Sur	D. P	Slack/ Sur	D. P	Slack/ Sur	D. P	Slack/ Sur	D. P
1.000	0.000	1.000	0.000	1.000	0.000	1.000	0.000	1.000	0.000	1.000
2.000	0.000	1.286	0.000	1.286	0.000	1.455	0.000	1.000	0.000	1.455
3.000	0.000	2.714	0.000	2.714	0.000	2.808	0.000	1.000	0.000	2.808
4.000	0.000	-1.000	0.000	-1.000	0.000	-1.000	0.000	-1.000	0.000	-1.000
5.000	0.000	44.143	0.000	44.143	0.000	47.313	0.000	45.000	0.000	47.313

Table 7: Optimal Table

	C		E		G		H		J	
Variable	Value	R. Cost	Value	R. Cost	Value	R. Cost	Value	R. Cost	Value	R. Cost
Z	1.000	0.000	1.000	0.000	1.000	0.000	1.000	0.000	1.000	0.000
L3	0.000	0.000	0.019	0.000	0.023	0.000	0.022	0.000	0.000	0.000
L5	1.000	0.000	0.340	0.000	0.409	0.000	0.391	0.000	1.000	0.000
L1	0.000	0.000	0.007	0.000	0.000	0.000	0.001	0.000	0.000	0.000
L2	0.000	0.000	0.004	0.000	0.009	0.000	0.006	0.000	0.000	0.000
L4	1.001	0.000	1.001	0.000	1.009	0.000	1.001	0.000	1.001	0.000
Row	Slack/ Surplus	D. Price	Slack/ Surplus	D. Price	Slack/ Surplus	D. Price	Slack/ Surplus	D. Price	Slack/ Surp	D. Price
1.000	1.001	0.000	0.736	0.000	0.886	0.000	0.848	0.000	1.000	0.000
2.000	1.000	0.000	0.019	0.000	0.023	0.000	0.022	0.000	0.999	0.000
3.000	1.000	0.000	0.226	0.000	0.273	0.000	0.261	0.000	1.000	0.000
4.000	1.000	0.000	0.679	0.000	0.818	0.000	0.783	0.000	1.000	0.000
5.000	1.001	0.000	1.000	0.000	1.205	0.000	1.152	0.000	1.001	0.000
6.000	1.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.999	0.000
7.000	1.001	0.000	0.830	0.000	1.000	0.000	0.957	0.000	1.000	0.000
8.000	1.001	0.000	0.868	0.000	1.045	0.000	1.000	0.000	1.001	0.000
9.000	1.000	0.000	0.132	0.000	0.159	0.000	0.152	0.000	0.999	0.000
10.000	1.000	0.000	0.528	0.000	0.636	0.000	0.609	0.000	1.000	0.000
11.000	0.000	1.000	0.000	1.000	0.000	1.000	0.000	1.000	0.000	1.000
12.000	0.000	0.000	0.211	0.000	0.159	0.000	0.165	0.000	0.001	0.000
13.000	0.001	0.000	0.917	0.000	1.077	0.000	1.019	0.000	0.002	0.000
14.000	0.000	-1.000	0.560	0.000	0.691	0.000	0.681	0.000	0.000	0.000
15.000	0.001	0.000	0.230	0.000	0.300	0.000	0.264	0.000	0.001	0.000
16.000	0.001	0.000	0.000	-1.000	0.077	0.000	0.000	0.000	0.002	0.000
17.000	0.001	0.000	0.871	0.000	0.991	0.000	0.970	0.000	0.001	0.000
18.000	0.000	0.000	0.081	0.000	0.000	-1.000	0.025	0.000	0.000	0.000
19.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	-1.000	0.000	0.000
20.000	0.001	0.000	0.759	0.000	0.905	0.000	0.862	0.000	0.001	0.000
21.000	0.000	0.000	0.337	0.000	0.327	0.000	0.345	0.000	0.000	-1.000
22.000	0.000	0.000	0.000	0.000	0.008	0.000	0.000	0.000	0.000	0.000
23.000	0.999	0.000	0.339	0.000	0.408	0.000	0.390	0.000	0.999	0.000

Table 8: Coefficients of the Model Parameters

	L3	L5	L1	L2	L4
E	**1.00000**	**18.000**	**0.3765**	**0.2258**	**53.003**
H	**1.00000**	**18.000**	**0.0503**	**0.2634**	**46.046**

Table 9: Transformed Data

	INPUT 1	INPUT 2	OUTPUT	SLOPE
A	30.71428	170.8571	39	
B	25.71428	176.8571	1	
C	13.71428	161.8571	12	0.073875
D	20.71428	178.8571	36	
E	22.71428	196.8571	53	**0.267459**
F	23.71428	164.8571	0	
G	28.71428	165.8571	44	0.261404
H	19.71428	170.8571	46	**0.267459**
I	21.71428	172.8571	7	
J	24.71428	161.8571	28	0.171012

Table 10: Outcome of IO CRS model

	C		E		G		H		J	
Variable	Value	Reduced Cost	Value	Reduced Cost	Value	Reduced Cost	Value	Reduced Cost	Value	Reduced Cost
T	0.375	0.000	**1.000**	0.000	0.985	0.000	**1.000**	0.000	0.643	0.000
S1	0.000	0.060	0.000	0.000	9.437	0.000	0.000	0.000	3.880	0.000
S2	16.125	0.000	0.000	0.005	0.000	0.005	0.000	0.005	0.000	0.005
S3	0.000	0.029	0.000	0.019	0.000	0.021	0.000	0.021	0.000	0.022
L1	0.000	0.882	0.000	0.133	0.000	0.166	0.000	0.163	0.000	0.171
L2	0.000	1.718	0.000	0.880	0.000	1.039	0.000	1.019	0.000	1.069
L3	0.000	0.641	0.000	0.595	0.000	0.695	0.000	0.681	0.000	0.715
L4	0.000	0.368	0.000	0.229	0.000	0.270	0.000	0.264	0.000	0.277
L5	0.000	0.000	1.000	0.000	0.000	0.000	0.000	0.000	0.528	0.000
L6	0.000	1.614	0.000	0.838	0.000	0.989	0.000	0.970	0.000	1.017
L7	0.000	0.605	0.000	0.013	0.000	0.024	0.000	0.024	0.000	0.024
L8	0.261	0.000	0.000	0.000	0.957	0.000	1.000	0.000	0.000	0.000
L9	0.000	1.291	0.000	0.746	0.000	0.879	0.000	0.861	0.000	0.904
L10	0.000	0.835	0.000	0.295	0.000	0.352	0.000	0.345	0.000	0.361
Row	Slack/ Surplus	D. Price	Slack/ Surplus	D. Price	Slack/ Surplus	D. Price	Slack/ Surplus	D. Price	Slack/ Surplus	D. Price
1.000	0.359	-1.000	1.000	-1.000	0.976	-1.000	1.000	-1.000	0.639	-1.000
2.000	0.000	0.061	0.000	0.000	0.000	0.001	0.000	0.001	0.000	0.001
3.000	0.000	0.001	0.000	0.005	0.000	0.006	0.000	0.006	0.000	0.006
4.000	0.000	-0.030	0.000	-0.019	0.000	-0.022	0.000	-0.022	0.000	-0.023