Optimization of economic functions with two variables

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Abstract

In addition to the mathematical notions for assigning stationary points with the help of the first partial derivatives of the two-variable function, we will classify these stationary points with the help of the second partial derivatives of the two-variable function, when the function f (x ,y) reaches the maximum and when it reaches the minimum. The above notions will be applied to the profit optimization of any enterprise that produces two goods. We will also optimize an enterprise that sells some kind of product in different markets with price discrimination.

Keywords: Function, derivative, optimization, product, profit

I. Introduction

1. Partial Derivatives

Based on [1],[3], we get:

Definition 1: A function f of the two variables x and y is a rule that assigns to each ordered pair (x,y) of real numbers in some set one and only one real number denoted by f (x,y)

Definition 2. Let z =f (x,y),

a. The first partial derivative of *f* with respect to x is:

$$\frac{\partial z}{\partial x} = f_x(x,y) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x} \qquad (1)$$

b. The first partial derivative of f with respect to y is:

$$\frac{\partial z}{\partial y} = f_y(x,y) = \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x,y)}{\Delta y} \qquad (2)$$

Computation of Partial Derivatives: The function $\frac{\partial z}{\partial x}$ or f_x is obtained by differentiating f with respect to x, treating y as a constant.

The function $\frac{\partial z}{\partial y}$ or f_y is obtained by differentiating f with respect to y, treating x as a constant.

Example 1. For the function $f(x,y)=5x^2-3xy+4y^2$, find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$?

Solution: Treating y as a constant, we obtain $\frac{\partial z}{\partial x}=10x-3y$, Treating x as a constant, we obtain

$\frac{\partial z}{\partial y}=-3x+8y$.

Example 2. Suppose that the production function $Q(x,y)=4000x^{0,5}y^{0,5}$ is known. Determine the marginal productivity of labor and the marginal productivity of capital when 25 units of labor and 169 units of capital are used.

Solution:

$$\frac{\partial Q}{\partial x}=4000\cdot 0{,}5\cdot x^{-0,5}y^{0,5}=\frac{2000y^{0,5}}{x^{0,5}}\ ,\ \frac{\partial Q}{\partial x}=4000\cdot 0{,}5\cdot x^{0,5}y^{-0,5}=\frac{2000x^{0,5}}{y^{0,5}}.$$

Substituting x = 25, and y = 169 , we obtain:

$$\left.\frac{\partial Q}{\partial x}\right|_{(25,169)}=\frac{2000\cdot 169^{0,5}}{25^{0,5}}=5200 \text{ units},$$

$$\left.\frac{\partial Q}{\partial y}\right|_{(25,169)}=\frac{2000\cdot 25^{0,5}}{169^{0,5}}=769{,}23 \text{ units}.$$

Thus we see that adding one unit of labor will increase production by about 5200 units and adding one unit of capital will increase production by about 769 units.

2. Second-Order Partial Derivatives

If z=f(x,y), the partial derivative of f_x with respect to x is:

$$f_{xx}=(f_x)_x \, or \, \frac{\partial^2 z}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) \qquad (3)$$

The partial derivative of f_x with respect to y is:

$$f_{xy}=(f_x)_y \, or \, \frac{\partial^2 z}{\partial y\partial x}=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) \qquad (4)$$

The partial derivative of f_y with respect to x is:

$$f_{yx} = \left(f_y\right)_x or \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) \qquad (5)$$

The partial derivative of f_y with respect to y is:

$$f_{yy} = \left(f_y\right)_y or \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) \qquad (6)$$

3. The Chain Rule; Approximation by the Total Differential

Recall that if z is a differentiable function of x and x is a differentiable function of t, then z can be regarded as a differentiable function of t and the rate of change of z with respect to t is given by the chain rule

$$\frac{dz}{dt} = \frac{dz}{dx}\frac{dx}{dt} \qquad (7)$$

Here is the corresponding rule for functions of two variables.

Suppose z is a function of x and y, each of which is a function of t then z can be regarded as a function of t and

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} \qquad (8)$$

Remark 1: $\frac{\partial z}{\partial x}\frac{dx}{dt}$ -rate of change of z with respect to t for fixed y

$\frac{\partial z}{\partial y}\frac{dy}{dt}$ -rate of change of z with respect to t for fixed x.

Example 3: A health store carries two kinds of multiple vitamins, Brand A and Brand B. Sales figures indicate that if Brand A is sold for x dollars per bottle and Brand B for y dollars per bottle, the demand for Brand A will be $Q(x, y) = 400 - 30x^2 + 30y$ bottles per month,

It is estimated that *t* months from now the price of Brand A will be $x = 3 + 0{,}05t$ dollars per bottle

and the price of Brand B will be $y = 3 + 0{,}1\sqrt{t}$ dollars per bottle.

At what rate will the demand for Brand A be changing with respect to time 5 months from now?

Solution: We goal is to find $\frac{dQ}{dt}$ when 5=4. Using the chain rule, we get

$$\frac{dQ}{dt}=\frac{\partial Q}{\partial x}\frac{dx}{dt}+\frac{\partial Q}{\partial y}\frac{dy}{dt}=-60x\cdot 0{,}05+20\cdot 0{,}05\cdot t^{-1/2}$$

When t=5, $x=3+0{,}05\cdot 5=3{,}25$ and hence

$$\frac{dQ}{dt}=-60\cdot 3{,}25\cdot 0{,}05+20\cdot 0{,}05\cdot 0{,}467=-9{,}3$$

That is, 5 months from now the monthly demand for Brand A will be decreasing at the rate of 9,3 bottles per month.

Let z = f (x, y) be a function defined in the set $D\subset R^2$ and M(x_1,y_1) a random poin from D. If to the variables x_1 and y_1 ,we give a change Δx_1 and Δy_1 respectively, so that the point $M'(x_1+\Delta x_1, y_1+\Delta y_1)$ stays in D, then the change of the function is

$$\Delta z = f(x_1+\Delta x_1, y_1+\Delta y_1)-f(x_1,y_1) \qquad (9)$$

Definition 3. Function f(x,y) is called differentiable at the point M(x_1,y_1), if the total change in that point can be presented in the following form.

$$\Delta z = A\Delta x_1 + B\Delta y_1 + \alpha\Delta x_1 + \beta\Delta y_1 \qquad (10)$$

Where A,B are real numbers that don’t depend from Δx_1 e Δy_1, whereas α,β are infinitely small when $\Delta x_1\to 0$, $\Delta y_1\to 0$, are equal to 0 when $\Delta x_1=\Delta y_1=0$.

The condition of differentiation can be written

$$\Delta z = A\Delta x_1 + B\Delta y_1 + O(\rho) \qquad (11)$$

where $O(\rho)=\alpha\Delta x_1+\beta\Delta y_1$

Theorema 1 ([6], p.176). If the function z=f(x,y) is differentiable at the point M(x_1,y_1), then at this point there are partial derivates and

$$\frac{\partial z}{\partial x_1}=A_1\ ,\ \frac{\partial z}{\partial y_1}=A_2 \qquad (12)$$

Consequence 1. The condition (2) of differentiation can be written in the following form:

$$\Delta z=\frac{\partial z}{\partial x_1}\Delta x_1+\frac{\partial z}{\partial y_1}\Delta y_1+O(\rho) \qquad (13)$$

Approximation Formula: Suppose z is a function of x and y. If Δx denotes a small change in x and Δy a small change in y, the corresponding change in z is

$$\Delta z \approx \frac{\partial z}{\partial x}\Delta x + \frac{\partial z}{\partial y}\Delta y \qquad (14)$$

The Total differential: If z is a function of x and y, the total differential of z is

$$dz = \frac{\partial z}{\partial x}\Delta x + \frac{\partial z}{\partial y}\Delta y \qquad (15)$$

Example 4: At a certain factory, the daily output is $Q = 60K^{1/2}L^{1/3}$ units, where K denotes the capital investment measured in units of \$ 1000 and L the size of the labor force measured in worker-hours. The current capital investment is \$ 900000 and 1000 and labor are used each day. Estimate the change in output that will result if capital investment is increased by \$1000 and labor is increased by 2 worker-hours.

Solution: Apply the approximation formula with K=900, L=1000, ΔK = 1,and ΔL= 2 to get

$$\Delta Q \approx \frac{\partial Q}{\partial K}\Delta K + \frac{\partial Q}{\partial L}\Delta L = 30K^{-1/2}L^{1/3}\Delta K + 20K^{1/2}L^{-2/3}\Delta L = 30\cdot\frac{1}{30}\cdot 10\cdot 1 + 20\cdot 30\cdot\frac{1}{100}\cdot 2 = 22 \text{ units.}$$

That is, output will increase by approximately 22 units.

II. Unconstrained optimization of functions with two variables

For the two variable function z = f(x,y) to be at a maximum or at a minimum, the first-order conditions which must be met are ∂z/∂x = 0 and ∂z /∂y = 0. To be at a maximum or minimum, the function must be at a stationary point with respect to changes in both variables. The second-order conditions and the reasons for them were relatively easy to explain in the case of a function of one independent variable. However, when two or more independent variables are involved the rationale for all the second-order conditions is not quite so straightforward. We shall therefore just state these second-order conditions here and give a brief intuitive explanation for the two-variable case before looking at some applications.

For the optimization of two variable functions there are two sets of second-order conditions. For any function z = f(x, y).

$$\frac{\partial^2 z}{\partial x^2} < 0 \text{ and } \frac{\partial^2 z}{\partial y^2} < 0 \text{ for a maximum}$$

$$\frac{\partial^2 z}{\partial x^2} > 0 \text{ and } \frac{\partial^2 z}{\partial y^2} > 0 \text{ for a minimum} \qquad (16)$$

These are similar to the second-order conditions for the optimization of a single variable function. The rate of change of a function (i.e. its slope) must be decreasing at a stationary point for that point to be a maximum and it must be increasing for a stationary point to be a minimum.

The difference here is that these conditions must hold with respect to changes in both independent variables.

The other second-order condition is

$$\left(\frac{\partial^2 z}{\partial x^2}\right)\cdot\left(\frac{\partial^2 z}{\partial y^2}\right)-\left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 > 0 \quad (17)$$

This must hold at both maximum and minimum stationary points.

Specifically, we use this algorithm for optimization:

Step 1. Solve the equations $f_x(x,y)=0$ and $f_y(x,y)=0$, to find the stationary point (a,b).

Step 2.

- If $f_{xx}>0$, $f_{yy}>0$ and $f_{xx}f_{yy}-f_{xy}^2>0$ in (a,b), then the function f(x,y) has a minimum in the point (a,b)
- If $f_{xx}<0$, $f_{yy}<0$ and $f_{xx}f_{yy}-f_{xy}^2>0$ in (a,b), then the function f(x,y) has a maximum in the point (a,b)
- If $f_{xx}f_{yy}-f_{xy}^2<0$ in (a,b), then the function has a saddle point in (a,b)

Example 5. A firm produces two products which are sold in two separate markets with the demand schedules $P_1 = 600 - 0{,}2Q_1$, $P_2 = 500 - 0{,}2Q_2$

Production costs are related and the firm faces the total cost function

$$TC = 16 + 1{,}2Q_1 + 1{,}5Q_2 + 0{,}2Q_1Q_2$$

If the firm wishes to maximize total profits, how much of each product should it sell? What will the maximum profit level be?

Solution. The total revenue is

$$TR = TR_1 + TR_2 = P_1Q_1 + P_2Q_2 = (600 - 0{,}3Q_1)Q_1 + (500 - 0{,}2Q_2)Q_2 =$$

$$=600Q_1 - 0{,}3\,Q_1^2 + 500Q_2 - 0{,}2\,Q_2^2$$

Therefore profit is

$$\pi = TR - TC = 600Q_1 - 0{,}3\,Q_1^2 + 500Q_2 - 0{,}2\,Q_2^2 - (16 + 1{,}2Q_1 + 1{,}5Q_2 + 0{,}2Q_1Q_2) =$$

$$=600Q_1 - 0{,}3\,Q_1^2 + 500Q_2 - 0{,}2\,Q_2^2 - 16 - 1{,}2Q_1 - 1{,}5Q_2 - 0{,}2Q_1Q_2 =$$

$$= -16 + 598{,}8Q_1 - 0{,}3\,Q_1^2 + 498{,}5Q_2 - 0{,}2\,Q_2^2 - 0{,}2Q_1Q_2$$

First-order conditions for maximization of this profit function are

$$\partial\pi / \partial Q_1 = 598{,}8 - 0{,}6Q_1 - 0{,}2Q_2 = 0 \qquad (18)$$

And

$$\partial\pi / \partial Q_2 = 498{,}5 - 0{,}4Q_2 - 0{,}2Q_1 = 0 \qquad (19)$$

Simultaneous equations (18) and (19) can now be solved to find the optimal values of q_1 and q_2. Multiplying (19) by 3 and Rearranging (18)

$$1495{,}5 - 1{,}2Q_2 - 0{,}6Q_1 = 0$$

$$598{,}8 - 0{,}2Q_2 - 0{,}6Q_1 = 0.$$

Subtracting gives Q_2=896,7 . Substituting this value for Q_2 into (18)

And we get Q_1=699,1

Checking second-order conditions by differentiating (18) and (19) again:

$$\frac{\partial^2\pi}{\partial Q_1^2} = -0{,}6 < 0\ , \quad \frac{\partial^2\pi}{\partial Q_2^2} = -0{,}4 < 0$$

This satisfies one set of second-order conditions for a maximum. The cross partial derivative will be

$$\frac{\partial^2\pi}{\partial Q_1 \partial Q_2} = -0{,}2$$

Therefore

$$\frac{\partial^2\pi}{\partial Q_1^2} \cdot \frac{\partial^2\pi}{\partial Q_2^2} = (-0{,}6)\cdot(-0{,}4) = 0{,}24 > (-0{,}2)^2 = \left(\frac{\partial^2\pi}{\partial Q_1 \partial Q_2}\right)^2$$

and so the remaining second-order condition for a maximum is satisfied. The actual profit is found by substituting the optimum values Q_1 = 699,1 and Q_2 = 896,7. into the profit function. Thus

$\pi = -16 + 598{,}8Q_1 - 0{,}3Q_1^2 + 498{,}5Q_2 - 0{,}2Q_2^2 - 0{,}2Q_1Q_2 = -16 + 598{,}8(699{,}1) - 0{,}3(699{,}1)\ 2 +$
$+498{,}5(896{,}7) - 0{,}2(896{,}7)\ 2 - 0{,}2(699{,}1)(896{,}7) = 432\ 797{,}02$

Example 6. A firm can sell its product in two countries, A and B, where demand in country A is given by $P_A = 100 - 2Q_A$ and in country B is $P_B = 100 - Q_B$.

It's total output is $Q_A + Q_B$, which it can produce at a cost of

$$TC = 50(Q_A+Q_B) + 0,5(Q_A+Q_B)^2$$

How much will it sell in the two countries assuming it maximises profits?

Solution:

$$\pi = TR - TC = P_AQ_A + P_BQ_B – TC$$

$$P_AQ_A = (100 – 2Q_A)Q_A$$

$$P_BQ_B = (100 – Q_B)\, Q_B$$

$$\pi = 100Q_A – 2Q_A^2 + 100Q_B – Q_B^2$$

$$– 50Q_A – 50Q_B – 0,5\,(Q_A+Q_B)^2$$

$$= 50Q_A – 2Q_A^2 + 50Q_B – Q_B^2 – 0,5\,(Q_A+Q_B)^2$$

Select Q_A and Q_B to max π:

If $\pi = 50Q_A – 2Q_A^2 + 50Q_B – Q_B^2 – 0,5\,(Q_A+Q_B)^2$

$$d\pi = 0$$

$$\pi_{QA} = 50 - 4Q_A – (0,5)\, 2\,(Q_A+Q_B) = 0$$

$$= 50 - 5Q_A – Q_B = 0 \qquad (20)$$

$$\pi_{QB} = 50 - 2Q_B – (0,5)2\,(Q_A+Q_B) = 0$$

$$= 50 - 3Q_B – Q_A = 0 \qquad (21)$$

$50 - 5Q_A – Q_B = 50 - 3Q_B – Q_A$

We take $2Q_A = Q_B$

Thus, output at stationary point is $(Q_A, Q_B) = (7^1/_7,\ 14\ ^2/_7)$

Check Sufficient conditions for Max: $d^2\pi < 0$

$$\pi_{QA} = 50 - 5Q_A – Q_B$$

$$\pi_{QB} = 50 - 3Q_B – Q_A$$

Then

$$\pi_{QAQA} = – 5 < 0$$

$$\pi_{QAQA}.\ \pi_{QBQB} – (\pi_{QAQB})^2 > 0$$

$$(-5)(-3)) - (-1)^2 = 14 > 0 \quad \text{Max}$$

So firm max profits by selling $7\frac{1}{7}$ units to country A and $14\frac{2}{7}$ units to country B.

Price discrimination is a microeconomic pricing strategy where identical or largely similar goods or services are transacted at different prices by the same provider in different markets. Price discrimination is distinguished from product differentiation by the more substantial difference in production cost for the differently priced products involved in the latter strategy . Price discrimination, very differently, relies on monopoly power, including market share, product uniqueness, sole pricing power, etc.

Example 7. A firm has set different prices for its family and industrial customers. If P_1 and Q_1 are the price and demand for the household market, then the demand equation is $P_1 + Q_1 = 500$. If P_2 and Q_2 are the price and demand for the industrial market, then the demand equation is

$2P_2 + 3Q_2 = 720$. The total cost function is

$TC = 50000 + 20Q$, where $Q = Q_1 + Q_2$.

Determine the firm's pricing policy that maximizes profit, with price discrimination, and calculate the value of the maximum price**.**

Solution:

For the family market, the demand equation is $P_1 = 500-Q_1$, while the function of total income is

$$TR_1 = P_1Q_1 = (500 - Q_1)Q_1 = 500Q_1 - Q_1^2.$$

For the industrial market, the demand equation is $P_2 = 360\text{-}1{,}5Q_2$, while the function of total revenue is

$$TR_2 = P_2Q_2 = (360 - \frac{3}{2}Q_2)Q_2 = 360Q_2 - 1{,}5Q_2^2.$$

For both markets we have

$$TR = TR_1 + TR_2 = P_2Q_2 = 500Q_1 - Q_1^2 + 360Q_2 - 1{,}5Q_2^2.$$

The total cost of producing these goods is given by $TC=50000+20Q$, where $Q=Q_1+Q_2$.

Therefore $TC = 50000 + 20(Q_1 + Q_2) = 50000 + 20Q_1 + 20Q_2$.

The profit function is

$$\pi = TR - TC = 500Q_1 - Q_1^2 + 360Q_2 - 1{,}5Q_2^2 - (50000 + 20Q_1 + 20Q) =$$

$$= 480Q_1 - Q_1^2 + 340Q_2 - 1{,}5Q_2^2 - 50000$$

This is a function of two variables Q_1 and Q_2, which we want to optimize. Partial derivatives of the first and second derivatives are:

$$\partial\pi / \partial Q_1 = 480 - 2Q_1 \,,\ \partial\pi / \partial Q_2 = 340 - 3Q_2.$$

While $\frac{\partial^2\pi}{\partial Q_1^2} = -2$, $\frac{\partial^2\pi}{\partial Q_2^2} = -3$, $\frac{\partial^2\pi}{\partial Q_1 \partial Q_2} = 0$

The two strategic steps are:

Step 1. At a stationary point we have $\partial\pi / \partial Q_1 = 0$ and $\partial\pi / \partial Q_2 = 0$, so we solve the equations

$480 - 2Q_1 = 0$ and $340 - 3Q_2 = 0$, from where we get $Q_1 = 240$, $Q_2 = 340/3$.

Step 2. It is easy to check whether the conditions for maximum are completed:

$$\frac{\partial^2\pi}{\partial Q_1^2} = -2 < 0,\quad \frac{\partial^2\pi}{\partial Q_2^2} = -3 < 0,\quad \frac{\partial^2\pi}{\partial Q_1^2} \cdot \frac{\partial^2\pi}{\partial Q_2^2} - \left(\frac{\partial^2\pi}{\partial Q_1 \partial Q_2} \right)^2 = (-2)(-3) - 0^2 = 6 > 0$$

Substituting $Q_1 = 240$, $Q_2 = 340/3$ in the corresponding demand equations, we obtain:

For the family market $P_1 = 500 - Q_1 = 500 - 240 = 260$ euros.

For the industrial market $P_2 = 360 - 3/2Q_2 = 360 - \frac{3}{2} \cdot \frac{340}{3} = 190$ euros.

By substituting $Q_1 = 240$, $Q_2 = 340/3$ at the profit function, we obtain:

$$\pi = 480Q_1 - Q_1^2 + 340Q_2 - 1{,}5Q_2^2 - 50000 = 26866{,}67 \text{ euros.}$$

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