Higher-Order Automated Theorem Provers

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Points to remember from this talk

1. Classical Higher-Order Logic (HOL): elegant, expressive, powerful
2. HOL-ATPs have recently made good progress
3. HOL is suited as a universal (meta-)logic
4. Cut-elimination is not a useful criterion in HOL

Talk Outline:

- Classical Higher-Order Logic (HOL)
- HOL-ATPs
- Some applications: Mathematics, Philosophy, AI
- HOL as universal (meta-)logic
- Cut-elimination versus cut-simulation
- Conclusion

Many important topics are not adressed here . . .

- Automation of Elementary Type Theory
- Higher-Order Unification, Pre-Unification, . . .
- Calculi: Resolution, Tableaux, Mating, . . .
- Skolemization
- Primitive Equality, Choice, Description, . . .
- Transformation(s) to FOL
- Proof formats
- . . .

More on such topics:

see the references in

[paper in APPA proceedings]

[BenzmüllerMiller, HandbookHistoryOfLogicVol.9, 2014 (to appear)]

Classical Higher-Order Logic (HOL)

(Church's Type Theory)

Classical Higher-Order Logic (HOL)

Expressivity	FOL	HOL	Example
Quantification over			
- Individuals	✓	✓	$\forall X\ p(f(X))$
- Functions	–	✓	$\forall F\ p(F(a))$
- Predicates/Sets/Rels	–	✓	$\forall P\ P(f(a))$
Unnamed			
- Functions	–	✓	$(\lambda X\ X)$
- Predicates/Sets/Rels	–	✓	$(\lambda X\ X \neq a)$
Statements about			
- Functions	–	✓	$\mathit{continuous}(\lambda X\ X)$
- Predicates/Sets/Rels	–	✓	$\mathit{reflexive}(=)$
Powerful abbreviations	–	✓	$\mathit{reflexive} = \lambda R\ \lambda X\ R(X, X)$

Expressivity	FOL	HOL	Example
Quantification over			
- Individuals	✓	✓	$\forall X_\iota\, p_{\iota\to o}(f_{\iota\to\iota}(X_\iota))$
- Functions	–	✓	$\forall F_{\iota\to\iota}\, p_{\iota\to o}(F_{\iota\to o}(a_\iota))$
- Predicates/Sets/Rels	–	✓	$\forall P_{\iota\to o}\, P_{\iota\to o}(f_{\iota\to\iota}(a_\iota))$
Unnamed			
- Functions	–	✓	$(\lambda X_\iota\, X_\iota)$
- Predicates/Sets/Rels	–	✓	$(\lambda X_{\iota\to\iota}\, X_{\iota\to\iota} \neq_{\iota\to\iota\to p} a)_\iota)$
Statements about			
- Functions	–	✓	$continuous_{(\iota\to\iota)\to o}(\lambda X_\iota\, X_\iota)$
- Predicates/Sets/Rels	–	✓	$reflexive_{(\iota\to\iota\to o)\to o}(=_{\iota\to\iota\to o})$
Powerful abbreviations	–	✓	$reflexive_{(\iota\to\iota\to o)\to o} = \lambda R_{(\iota\to\iota\to o)}\, \lambda X_\iota\, R(X,X)$

Simple Types: Prevent Paradoxes and Inconsistencies

Classical Higher-Order Logic (HOL) / Church's Simple Type Theory

- Simple Types $\alpha ::= \iota \mid o \mid \alpha_1 \rightarrow \alpha_2$

- Simple Types $\alpha ::= \iota \mid o \mid \alpha_1 \rightarrow \alpha_2$

  Individuals → ι

  Booleans (True and False) → o

  Functions → $\alpha_1 \rightarrow \alpha_2$

- Simple Types $\alpha ::= \mu \mid \iota \mid o \mid \alpha_1 \to \alpha_2$

Possible worlds → μ

Individuals → ι

Booleans (True and False) → o

Functions → $\alpha_1 \to \alpha_2$

Classical Higher-Order Logic (HOL) / Church's Simple Type Theory

- Simple Types $\alpha ::= \mu \mid \iota \mid o \mid \alpha_1 \to \alpha_2$
- HOL Syntax

$$
\begin{aligned}
s, t \quad ::= \quad & c_\alpha \mid X_\alpha \mid (\lambda X_\alpha\, s_\beta)_{\alpha\to\beta} \mid (s_{\alpha\to\beta}\, t_\alpha)_\beta \\
& \mid (\neg_{o\to o}\, s_o)_o \mid (s_o \vee_{o\to o\to o} t_o)_o \mid (\forall X_\alpha\, t_o)_o
\end{aligned}
$$

Constant Symbols
Variable Symbols

Classical Higher-Order Logic (HOL) / Church's Simple Type Theory

- Simple Types $\alpha ::= \mu \mid \iota \mid o \mid \alpha_1 \rightarrow \alpha_2$
- HOL Syntax

$$s, t \quad ::= \quad c_\alpha \mid X_\alpha \mid (\lambda X_\alpha\, s_\beta)_{\alpha\rightarrow\beta} \mid (s_{\alpha\rightarrow\beta}\, t_\alpha)_\beta$$
$$\mid (\neg_{o\rightarrow o}\, s_o)_o \mid (s_o \vee_{o\rightarrow o\rightarrow o}\, t_o)_o \mid (\forall X_\alpha\, t_o)_o$$

Constant Symbols
Variable Symbols
Abstraction
Application

Classical Higher-Order Logic (HOL) / Church's Simple Type Theory

- Simple Types $\alpha ::= \mu \mid \iota \mid o \mid \alpha_1 \rightarrow \alpha_2$
- HOL Syntax

$$\begin{aligned} s,t \quad ::= \quad & c_\alpha \mid X_\alpha \mid (\lambda X_\alpha\, s_\beta)_{\alpha\rightarrow\beta} \mid (s_{\alpha\rightarrow\beta}\, t_\alpha)_\beta \\ & \mid (\neg_{o\rightarrow o}\, s_o)_o \mid (s_o \vee_{o\rightarrow o\rightarrow o} t_o)_o \mid (\forall X_\alpha\, t_o)_o \end{aligned}$$

Constant Symbols
Variable Symbols
Abstraction
Application
Logical Connectives

- Simple Types $\alpha ::= \mu \mid \iota \mid o \mid \alpha_1 \to \alpha_2$
- HOL Syntax

$$s, t \quad ::= \quad c_\alpha \mid X_\alpha \mid (\lambda X_\alpha\, s_\beta)_{\alpha\to\beta} \mid (s_{\alpha\to\beta}\, t_\alpha)_\beta$$
$$\mid (\neg_{o\to o}\, s_o)_o \mid (s_o \vee_{o\to o\to o} t_o)_o \mid \underbrace{(\forall X_\alpha\, t_o)_o}_{(\Pi_{(\alpha\to o)\to o}\,(\lambda X_\alpha\, t_o))_o}$$

Classical Higher-Order Logic (HOL) / Church's Simple Type Theory

- Simple Types $\alpha ::= \mu \mid \iota \mid o \mid \alpha_1 \rightarrow \alpha_2$
- HOL Syntax

$$
\begin{array}{rcl}
s, t & ::= & c_\alpha \mid X_\alpha \mid (\lambda X_\alpha\, s_\beta)_{\alpha\rightarrow\beta} \mid (s_{\alpha\rightarrow\beta}\, t_\alpha)_\beta \\
 & & \mid (\neg_{o\rightarrow o}\, s_o)_o \mid (s_o \vee_{o\rightarrow o\rightarrow o} t_o)_o \mid (\Pi_{(\alpha\rightarrow o)\rightarrow o}\, (\lambda X_\alpha\, t_o))_o
\end{array}
$$

- Terms of type o: formulas

Classical Higher-Order Logic (HOL) / Church's Simple Type Theory

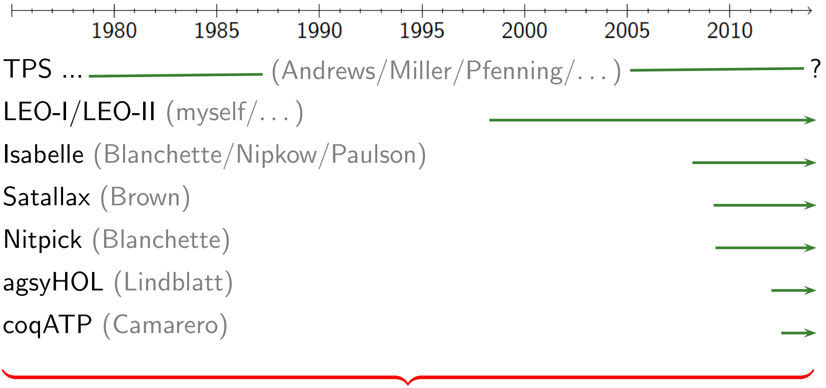
- Simple Types $\alpha ::= \mu \mid \iota \mid o \mid \alpha_1 \rightarrow \alpha_2$
- HOL Syntax

$$s,t ::= c_\alpha \mid X_\alpha \mid (\lambda X_\alpha\, s_\beta)_{\alpha\rightarrow\beta} \mid (s_{\alpha\rightarrow\beta}\, t_\alpha)_\beta$$
$$\mid (\neg_{o\rightarrow o}\, s_o)_o \mid (s_o \vee_{o\rightarrow o\rightarrow o}\, t_o)_o \mid (\Pi_{(\alpha\rightarrow o)\rightarrow o}\,(\lambda X_\alpha\, t_o))_o$$

- Terms of type o: formulas
- HOL is (meanwhile) well understood
  - Origin [Church, J.Symb.Log., 1940]
  - Henkin-Semantics [Henkin, J.Symb.Log., 1950]
    [Andrews, J.Symb.Log., 1971, 1972]
  - Extens./Intens. [BenzmüllerEtAl., J.Symb.Log., 2004]
    [Muskens, J.Symb.Log., 2007]
- HOL with Henkin-Semantics: semi-decidable & compact (like FOL)

Higher-Order Automated Theorem Provers (HOL-ATPs)

HOL-ATPs



- all accept TPTP THF0 syntax
- can be called remotely via SystemOnTPTP at Miami
- they significantly gained in strength over the last years
- they can be bundled into a combined prover HOL-P

HOL-ATPs

EU FP7 Project THFTPTP

- Collaboration with Geoff Sutcliffe and others (Chad Brown, Florian Rabe, Nik Sultana, Jasmin Blanchette, Frank Theiss, . . .)
- Results
  - THF0 syntax for HOL (with Choice; Henkin Semantics)
  - library with example problems (e.g. entire TPS library) and results
  - international CASC competition for HOL-ATP
  - online access to provers
  - various tools

More information: [SutcliffeBenzmüller, J.FormalizedReasoning, 2010]

http://cordis.europa.eu/result/report/rcn/45614_en.html

- **2009:** Winner TPS
- **2010:** Winner LEO-II 1.2 solved 56% more (than previous winner)
- **2011:** Winner Satallax 2.1 solved 21% more
- **2012:** Winner Isabelle-HOT-2012 solved 35% more
- **2013:** Winner Satallax-MaLeS solved 21% more

Some Applications in
Mathematics & Philosophy & AI

Some Applications: Mathematics

ATPs as external reasoners in Interactive Proof Assistants

[KaliszykUrban, Learning-Assisted Automated Reasoning with Flyspeck, JAR, 2014]

- Flyspeck project: formal proof (in HOL-light) of Kepler's Conjecture
- automation of 14185 theorems studied by Kaliszyk and Urban
- they developed AI architecture employing various external ATPs in which 39 % of the theorems could be proved in a push-button mode in 30 seconds of real time on a fourteen-CPU workstation
- subset of 1419 theorems extracted from Flyspeck theorems
- **next slide:** performance of THF0 provers on these 1419 problems

Some Applications: Mathematics

196 C. Kaliszyk, J. Urban

Table 7 All ATP re-proving with 30s time limit on 10 % of problems

Prover	Theorem (%)	Unique	SOTAC	Σ-SOTAC	CounterSat (%)	Processed
Isabelle	587 (41.3)	39	0.201	118.09	0 (0.0)	1419
Epar	545 (38.4)	9	0.131	71.18	0 (0.0)	1419
Z3	513 (36.1)	17	0.149	76.49	0 (0.0)	1419
E 1.6	463 (32.6)	0	0.101	46.69	0 (0.0)	1419
LEO2-po1	441 (31.0)	1	0.106	46.85	0 (0.0)	1419
Vampire	434 (30.5)	3	0.107	46.44	0 (0.0)	1419
CVC3	411 (28.9)	4	0.111	45.76	0 (0.0)	1419
Satallax	383 (26.9)	7	0.130	49.69	1 (0.0)	1419
Yices	360 (25.3)	0	0.097	35.06	0 (0.0)	1419
iProver	348 (24.5)	0	0.088	30.50	9 (0.6)	1419
Prover9	345 (24.3)	0	0.087	30.07	0 (0.0)	1419
Metis	331 (23.3)	0	0.085	28.23	0 (0.0)	1419
SPASS	326 (22.9)	0	0.081	26.46	0 (0.0)	1419
leanCoP	305 (21.4)	1	0.092	27.96	0 (0.0)	1419

Some Applications: Philosophy

Theoretical Philosophy and Metaphysics

[Benzmüller&Woltzenlogel-Paleo, AutomatingGödel'sOntologicalProof, ECAI, 2014]

- First-time verification/automation of a modern ontological argument

Gödel's/Scott's proof of the existence of God

- Remember Leibniz: Two debating philosophers . . . Calculemus!
- Gödel's argument employs Higher-Order Modal Logic

See also the talk by:

Bruno Woltzenlogel-Paleo, NCPROOFS WS, July 20, 12:15 (FH, SR104)

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Nachrichten > Wissenschaft > Mensch > Mathematik > Formel von Kurt Gödel: Mathematiker bestätigen Gottesbeweis

Formel von Kurt Gödel: Mathematiker bestätigen Gottesbeweis

Von Tobias Hürter

picture-alliance/ Imagno/ Wiener Stadt- und Landesbibliothek

Kurt Gödel (um das Jahr 1935): Der Mathematiker hielt seinen Gottesbeweis jahrzehntelang geheim

Ein Wesen existiert, das alle positiven Eigenschaften in sich vereint. Das bewies der legendäre Mathematiker Kurt Gödel mit einem komplizierten Formelgebilde. Zwei Wissenschaftler haben diesen Gottesbeweis nun überprüft - und für gültig befunden.

Montag, 09.09.2013 – 12:03 Uhr

Drucken | Versenden | Merken

Jetzt sind die letzten Zweifel ausgeräumt: Gott existiert tatsächlich. Ein Computer hat es mit kalter Logik bewiesen - das MacBook des Computerwissenschaftlers Christoph Benzmüller von der Freien Universität Berlin.

Austria
- Die Presse
- Wiener Zeitung
- ORF
- . . .

Italy
- Repubblica
- Ilsussidario
- . . .

India
- Delhi Daily News
- India Today
- . . .

US
- ABC News
- . . .

International
- Spiegel International
- United Press Intl.
- . . .

Germany
- Telepolis & Heise
- Spiegel Online
- FAZ
- Die Welt
- Berliner Morgenpost
- . . .

Many more links at: https://github.com/FormalTheology/GoedelGod


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English Site > Germany > Science > Scientists Use Computer to Mathematically Prove Gödel God Theorem

Holy Logic: Computer Scientists 'Prove' God Exists

By David Knight

picture-alliance/ Imagno/ Wiener Stadt- und Landesbibliothek

Austrian mathematician Kurt Gödel kept his proof of God's existence a secret for decades. Now two scientists say they have proven it mathematically using a computer.

Two scientists have formalized a theorem regarding the existence of God penned by mathematician Kurt Gödel. But the God angle is somewhat of a red herring -- the real step forward is the example it sets of how computers can make scientific progress simpler.

Germany
- Telepolis & Heise
- Spiegel Online
- FAZ
- Die Welt
- Berliner Morgenpost
- ...

Austria
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- ...

India
- Delhi Daily News
- India Today
- ...

US
- ABC News
- ...

International
- Spiegel International
- United Press Intl.
- ...

Many more links at: `https://github.com/FormalTheology/GoedelGod`

Some Applications: Artificial Intelligence

Quantified Conditional Logics (QCLs)

[Benzmüller, AutomatingQuantifiedConditionalLogicsInHOL, IJCAI, 2013]

- known as logics of normality or typicality
- many applications: action planning, counterfactual reasoning, default reasoning, deontic reasoning, reasoning about knowledge, ...
- examples [Delgrande, Artif.Intell., 1998]:
  "Birds normally fly, penguins normally do not fly and all penguins are necessarily birds."
- not yet widely studied
- no direct provers implemented so far

- automation of QCLs possible in HOL (via semantic embedding)
- cut-elimination as a side result

HOL as a Universal (Meta-)Logic:
Quantified Conditional Logics (QCLs)

QCLs are fragments of HOL

Syntax

$$\varphi, \psi ::= \quad P \mid k(X^1, \ldots, X^n) \mid \neg\varphi \mid \varphi \vee \psi \mid \varphi \Rightarrow \psi \mid$$
$$\forall^{co} X \varphi \mid \forall^{va} X \varphi \mid \forall P \varphi$$

conditional operator

Kripke style semantics

$M, g, s \models \varphi \vee \psi$ iff $M, g, s \models \varphi$ or $M, g, s \models \psi$

$\ldots\ldots$

$M, g, s \models \varphi \Rightarrow \psi$ iff $M, g, t \models \psi$ for all $t \in S$ such that $t \in f(s, [\varphi])$ where $[\varphi] = \{u \mid M, g, u \models \varphi\}$

$\ldots\ldots$

Selection function [Stalnaker, 1968]
(cf. accessibility relations in modal logics)

QCLs are fragments of HOL

QCL formulas φ are identified with (lifted) HOL terms φ_τ where $\tau := \iota \to o$

Semantic embedding exploits Kripke style semantics

$$
\begin{aligned}
\neg &= \lambda A_\tau \lambda X_\iota \neg (A\, X) \\
\vee &= \lambda A_\tau \lambda B_\tau \lambda X_\iota (A\, X \vee B\, X) \\
\Rightarrow &= \lambda A_\tau \lambda B_\tau \lambda X_\iota \forall V_\iota (f\, X\, A\, V \to B\, V) \\
\forall^{co} &= \lambda Q_{u\to\tau} \lambda V_\iota \forall X_u (Q\, X\, V) \\
\forall^{va} &= \lambda Q_{u\to\tau} \lambda V_\iota \forall X_u (eiw\, V\, X \to Q\, X\, V) \\
\forall &= \lambda R_{\tau\to\tau} \lambda V_\iota \forall P_\tau (R\, P\, V)
\end{aligned}
$$

Meta-notion of validity defined as: $\text{valid} = \lambda A_\tau \forall S_\iota (A\, S)$

Varying domains are non-empty: $\forall W_\iota \exists X_u (eiw\, W\, X)$

A very „lean" QCL Theorem Prover (in HOL)

```
%---- file: Axioms.ax ----------------------------------
%--- type mu for individuals
thf(mu,type,(mu:$tType)).
%--- reserved constant for selection function f
thf(f,type,(f:$i>($i>$o)>$i>$o)).
%--- `exists in world' predicate for varying domains;
%--- for each v we get a non-empty subdomain eiw@v
thf(eiw,type,(eiw:$i>mu>$o)).
thf(nonempty,axiom,(![V:$i]:?[X:mu]:(eiw@V@X))).
%--- negation, disjunction, material implication
thf(not,type,(not:($i>$o)>$i>$o)).
thf(or,type,(or:($i>$o)>($i>$o)>$i>$o)).
thf(not_def,definition,(not = (^[A:$i>$o,X:$i]:~(A@X)))).
thf(or_def,definition,(or = (^[A:$i>$o,B:$i>$o,X:$i]:((A@X)|(B@X))))).
%--- conditionality
thf(cond,type,(cond:($i>$o)>($i>$o)>$i>$o)).
thf(cond_def,definition,(cond = (^[A:$i>$o,B:$i>$o,X:$i]:![W:$i]:((f@X@A@W)=>(B@W))))).
%--- quantification (constant dom., varying dom., prop.)
thf(all_co,type,(all_co: (mu>$i>$o)>$i>$o)).
thf(all_va,type,(all_va:(mu>$i>$o)>$i>$o)).
thf(all,type,(all:(($i>$o)>$i>$o)>$i>$o)).
thf(all_co_def,definition,(all_co = (^[A:mu>$i>$o,W:$i]:![X:mu]:(A@X@W)))).
thf(all_va_def,definition,(all_va = (^[A:mu>$i>$o,W:$i]:![X:mu]:((eiw@W@X)=>(A@X@W))))).
thf(all_def,definition,(all = (^[A:($i>$o)>$i>$o,W:$i]:![P:$i>$o]:(A@P@W)))).
%--- notion of validity of a conditional logic formula
thf(vld,type,(vld:($i>$o)>$o)).
thf(vld_def,definition,(vld = (^[A:$i>$o]:![S:$i]:(A@S)))).
%---- end file: Axioms.ax ------------------------------
```

QCLs are fragments of HOL

Theorem (Soundness and Completeness [Benzmüller, IJCAI, 2013])

$$\models^{QCL} \varphi \quad \textit{iff} \quad \models^{HOL} \textit{valid}\, \varphi_\tau$$

Soundness and Completeness Results for Various Logics

$$\models^{L} \varphi \quad \text{iff} \quad \models^{HOL} \text{valid}\, \varphi_{\tau}$$

- Prop. Multimodal Logics [BenzmüllerPaulson, Log.J.IGPL, 2010]
- Quantified Multimodal Logics [BenzmüllerPaulson, Logica Universalis, 2012]
- Higher-Order Multimodal Logics [BenzmüllerWoltzenlogelP., ECAI, 2014]
- Prop. Conditional Logics [BenzmüllerGenoveseGabbayRispoli, AMAI, 2012]
- Quantified Conditional Logics [Benzmüller, IJCAI, 2013]
- Intuitionistic Logics: [BenzmüllerPaulson, Log.J.IGPL, 2010]
- Access Control Logics: [Benzmüller, IFIP SEC, 2009]
- Combinations of Logics: [Benzmüller, AMAI, 2011]

Cut-Elimination versus Cut-Simulation

Cut-Elimination versus Cut-Simulation

[BenzmüllerBrownKohlhase, Cut-Simulation in Impredicative Logics, LMCS, 2009]

- studies Henkin complete, one-sided sequent calculi for HOL
- cut-elimination proved for a 'naive' calculus
- cut-simulation shown for this calculus
- improved calculi presented that avoid cut-simulation effects

- Why relevant?

  Ideas of the improved calculi are also present in
  LEO-II (resolution) and Satallax (tableaux)

One-sided Sequent Calculus G1

Δ and Δ': finite sets of β-normal closed formulas
$\Delta, \mathbf{A}$ stands for $\Delta \cup \{\mathbf{A}\}$
$l \doteq r$ denotes Leibniz equality: $\Pi(\lambda P_{\alpha\to o}(\neg Pl \vee Pr))$

Basic Rules

$$\frac{\Delta, s}{\Delta, \neg\neg s}\,\mathcal{G}(\neg) \qquad \frac{\Delta, \neg s \quad \Delta, \neg t}{\Delta, \neg(s \vee t)}\,\mathcal{G}(\vee_-) \qquad \frac{\Delta, s, t}{\Delta, (s \vee t)}\,\mathcal{G}(\vee_+)$$

$$\frac{\Delta, \neg\, (sl){\downarrow_\beta} \quad l_\alpha \text{ closed term}}{\Delta, \neg\Pi^\alpha s}\,\mathcal{G}(\Pi^l_-) \qquad \frac{\Delta, (sc){\downarrow_\beta} \quad c_\delta \text{ new symbol}}{\Delta, \Pi^\alpha s}\,\mathcal{G}(\Pi^c_+)$$

Initialization

$$\frac{s \text{ atomic (and } \beta\text{-normal)}}{\Delta, s, \neg s}\,\mathcal{G}(init)$$

One-sided Sequent Calculus G1

Δ and Δ': finite sets of β-normal closed formulas
$\Delta, \mathbf{A}$ stands for $\Delta \cup \{\mathbf{A}\}$
$l \doteq r$ denotes Leibniz equality: $\Pi(\lambda P_{\alpha\to o}(\neg Pl \vee Pr))$

Basic Rules

$$\frac{\Delta, s}{\Delta, \neg\neg s}\,\mathcal{G}(\neg) \qquad \frac{\Delta, \neg s \quad \Delta, \neg t}{\Delta, \neg(s \vee t)}\,\mathcal{G}(\vee_-) \qquad \frac{\Delta, s, t}{\Delta, (s \vee t)}\,\mathcal{G}(\vee_+)$$

$$\frac{\Delta, \neg\,(sl){\downarrow_\beta} \quad l_\alpha \text{ closed term}}{\Delta, \neg\Pi^\alpha s}\,\mathcal{G}(\Pi^l_-) \qquad \frac{\Delta, (sc){\downarrow_\beta} \quad c_\delta \text{ new symbol}}{\Delta, \Pi^\alpha s}\,\mathcal{G}(\Pi^c_+)$$

Initialization

$$\frac{s \text{ atomic (and } \beta\text{-normal)}}{\Delta, s, \neg s}\,\mathcal{G}(init)$$

Boolean extensionality axiom ($\mathcal{B}_o$)

$$\forall A_o \forall B_o((A \longleftrightarrow B) \to A \doteq^o B)$$

$$\frac{\Delta, \neg\mathcal{B}_o}{\Delta}\,\mathcal{G}(\mathcal{B})$$

Infinitely many functional extensionality axioms ($\mathcal{F}_{\alpha\beta}$)

$$\forall F_{\alpha\to\beta} \forall G_{\alpha\to\beta}(\forall X_\alpha(FX \doteq^\beta GX) \to F \doteq^{\alpha\to\beta} G)$$

$$\frac{\Delta, \neg\mathcal{F}_{\alpha\beta} \quad \alpha \to \beta \in \mathcal{T}}{\Delta}\,\mathcal{G}(\mathcal{F}_{\alpha\beta})$$

Theorem (Soundness/Completeness [BenzmüllerBrownKohlhase, LMCS, 2009])

G1 is sound and complete for HOL: $\models^{HOL} s$ *iff* $\vdash^{G1} s$

Theorem (Cut-elimination [BenzmüllerBrownKohlhase, LMCS, 2009])

The rule $\mathcal{G}(cut)$

$$\frac{\Delta, s \quad \Delta, \neg s}{\Delta}\ \mathcal{G}(cut)$$

is admissible in G1.

But: G1 supports effective simulation of the cut-rule!

In other words: the above cut-elimination result is meaningless.

One-sided Sequent Calculus $G1$

Cut-simulation with the Boolean extensionality axiom

$$
\dfrac{
  \dfrac{
    \dfrac{\text{derivable in 7 steps} \atop \vdots}{\Delta, a \longleftrightarrow a}
  }{\Delta, \neg\neg(a \longleftrightarrow a)} \; \mathcal{G}(\neg)
  \qquad
  \dfrac{\Delta, s \quad \Delta, \neg s}{\vdots \;\; \text{derivable in 3 steps, see below}}
  \atop
  \Delta, \neg(a \doteq^{o} a)
}{
  \dfrac{\Delta, \neg(\neg(a \longleftrightarrow a) \vee a \doteq^{o} a)}{\Delta, \neg\mathcal{B}_o} \; 2 \times \mathcal{G}(\Pi^{a}_{-})
} \; \mathcal{G}(\vee_{-})
$$

Cut-simulation with the Boolean extensionality axiom

$$\dfrac{\dfrac{\dfrac{\dfrac{\overset{\text{derivable in 7 steps}}{\vdots}}{\Delta, a \longleftrightarrow a}}{\Delta, \neg\neg(a \longleftrightarrow a)}\;\mathcal{G}(\neg) \qquad \dfrac{\Delta, s \quad \Delta, \neg s}{\Delta, \neg(a \doteq^o a)}\;\vdots\ \text{derivable in 3 steps, see below}}{\Delta, \neg(\neg(a \longleftrightarrow a) \vee a \doteq^o a)}\;\mathcal{G}(\vee_-)}{\Delta, \neg\mathcal{B}_o}\;2 \times \mathcal{G}(\Pi^a_-)$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\Delta, s}{\Delta, \neg\neg s}\;\mathcal{G}(\neg) \qquad \Delta, \neg s}{\Delta, \neg(\neg s \vee s)}\;\mathcal{G}(\vee_-)}{\Delta, \neg\forall P_{\alpha\to o}(\neg Pa \vee Pa)}\;\mathcal{G}(\Pi^{\lambda X\, s}_-)}{\Delta, \neg(a \doteq^o a)}\;def.$$

Cut-simulation with functional extensionality axiom

derivable in 3 steps

$$\vdots$$

$$\dfrac{\dfrac{\Delta, fb \doteq^{\beta} fb}{\Delta, (\forall X_\alpha fX \doteq^{\beta} fX)}\ \mathcal{G}(\Pi^b_+)}{\Delta, \neg\neg\forall X_\alpha fX \doteq^{\beta} fX}\ \mathcal{G}(\neg)$$

$$\Delta, s \qquad \Delta, \neg s$$

$\vdots$ derivable in 3 steps

$$\Delta, \neg(f \doteq^{\alpha\to\beta} f)$$

$$\dfrac{\dfrac{\Delta, \neg\neg\forall X_\alpha fX \doteq^{\beta} fX \qquad \Delta, \neg(f \doteq^{\alpha\to\beta} f)}{\Delta, \neg(\neg(\forall X_\alpha fX \doteq^{\beta} fX) \vee f \doteq^{\alpha\to\beta} f)}\ \mathcal{G}(\vee_-)}{\Delta, \neg\mathcal{F}_{\alpha\beta}}\ 2 \times \mathcal{G}(\Pi^f_-)$$

One-sided Sequent Calculus G2

Basic Rules

$$\frac{\Delta, s}{\Delta, \neg\neg s}\,\mathcal{G}(\neg) \qquad \frac{\Delta, \neg s \quad \Delta, \neg t}{\Delta, \neg(s \lor t)}\,\mathcal{G}(\lor_-) \qquad \frac{\Delta, s, t}{\Delta, (s \lor t)}\,\mathcal{G}(\lor_+)$$

$$\frac{\Delta, \neg\ (sl)\big\downarrow_\beta \quad l_\alpha \text{ closed term}}{\Delta, \neg\Pi^\alpha s}\,\mathcal{G}(\Pi^l_-) \qquad \frac{\Delta,\ (sc)\big\downarrow_\beta \quad c_\delta \text{ new symbol}}{\Delta, \Pi^\alpha s}\,\mathcal{G}(\Pi^c_+)$$

Initialization

$$\frac{s \text{ atomic (and } \beta\text{-normal)}}{\Delta, s, \neg s}\,\mathcal{G}(init) \qquad \frac{\Delta, (s \doteq^o t) \quad s,t \text{ atomic}}{\Delta, \neg s, t}\,\mathcal{G}(Init^{\doteq})$$

Extensionality Rules

$$\frac{\Delta,\ (\forall X_\alpha sX \doteq^\beta tX)\big\downarrow_\beta}{\Delta, (s \doteq^{\alpha\to\beta} t)}\,\mathcal{G}(\mathfrak{f}) \qquad \frac{\Delta, \neg s, t \quad \Delta, \neg t, s}{\Delta, (s \doteq^o t)}\,\mathcal{G}(\mathfrak{b})$$

$$\frac{\Delta, (s^1 \doteq^{\alpha_1} t^1) \ \cdots\ \Delta, (s^n \doteq^{\alpha_n} t^n) \qquad n \geq 1, \beta \in \{o, \iota\},\ h_{\overline{\alpha^n}\to\beta} \in \Sigma}{\Delta, (h\overline{s^n} \doteq^\beta h\overline{t^n})}\,\mathcal{G}(d)$$

Cut-Simulation with Prominent Axioms

- Axiom of excluded middle — 3 steps
- Instances of the comprehension axioms — 16 steps
- Leibniz equations (axioms/hypotheses) — 3 steps
- Reflexivity definition of equality (Andrews) — 4 steps
- Axiom of functional extensionality — 11 steps
- Axiom of Boolean extensionality — 14 steps
- Axioms of choice — 7 steps
- Axiom of description — 25 steps
- Axiom of Iinduction — 18 steps

Consequence: HOL-ATPs should better avoid these axioms!

Cut-Elimination for QCL

We have

Theorem (Soundness and Completeness of QCL Embedding in HOL)

$$\models^{QCL} \varphi \quad \textit{iff} \quad \models^{HOL} \textit{valid}\, \varphi_\tau$$

Theorem (Soundness and Completeness of HOL)

$$\models^{HOL} \varphi \ \textit{iff} \ \vdash^{G1/G2}_{cut-free} \varphi$$

Putting things together

Theorem (Sound and Complete Cut-free Calculi for QCL)

$$\models^{QCL} \varphi \quad \textit{iff} \quad \vdash^{G1/G2}_{cut-free} \textit{valid}\, \varphi_\tau$$

Thus, we obtain a cut-elimination result for QCLs (and many, many other non-classical logics) for free!
(But due to cut-simulation effects these results could be meaningless.)

Conclusion

Points to remember from this talk

1. Classical Higher-Order Logic (HOL): elegant, expressive, powerful
2. HOL-ATPs have recently made good progress
3. HOL is suited as a universal (meta-)logic
4. Cut-elimination is not a useful criterion in HOL

Remember: many relevant topics have not been adressed ...

- Automation of Elementary Type Theory
- Higher-Order Unification, Pre-Unification, ...
- Calculi: Resolution, Tableaux, Mating, ...
- Skolemization
- Primitive Equality, Choice, Description, ...
- Transformation(s) to FOL
- ...

QCLs are fragments of HOL

ID	Syn. Axiom Sem. Condition	$A \Rightarrow A$ $f(w, [A]) \subseteq [A]$
MP	Syn. Axiom Sem. Condition	$(A \Rightarrow B) \rightarrow (A \rightarrow B)$ $w \in [A] \rightarrow w \in f(w, [A])$
CS	Syn. Axiom Sem. Condition	$(A \wedge B) \rightarrow (A \Rightarrow B)$ $w \in [A] \rightarrow f(w, [A]) \subseteq \{w\}$
CEM	Syn. Axiom Sem. Condition	$(A \Rightarrow B) \vee (A \Rightarrow \neg B)$ $\lvert f(w, [A]) \rvert \leq 1$
AC	Syn. Axiom Sem. Condition	$(A \Rightarrow B) \wedge (A \Rightarrow C) \rightarrow (A \wedge C \Rightarrow B)$ $f(w, [A]) \subseteq [B] \rightarrow f(w, [A \wedge B]) \subseteq f(w, [A])$
RT	Syn. Axiom Sem. Condition	$(A \wedge B \Rightarrow C) \rightarrow ((A \Rightarrow B) \rightarrow (A \Rightarrow C))$ $f(w, [A]) \subseteq [B] \rightarrow f(w, [A]) \subseteq f(w, [A \wedge B])$
CV	Syn. Axiom Sem. Condition	$(A \Rightarrow B) \wedge \neg(A \Rightarrow \neg C) \rightarrow (A \wedge C \Rightarrow B)$ $(f(w, [A]) \subseteq [B]$ and $f(w, [A]) \cap [C] \neq \emptyset) \rightarrow f(w, [A \wedge C]) \subseteq [B]$
CA	Syn. Axiom Sem. Condition	$(A \Rightarrow B) \wedge (C \Rightarrow B) \rightarrow (A \vee C \Rightarrow B)$ $f(w, [A \vee B]) \subseteq f(w, [A]) \cup f(w, [B])$

QCLs are fragments of HOL

For automating logic ID with HOL-ATPs simply add

$$\text{valid } \Pi\lambda A.A \Rightarrow A$$

or

$$(\forall A, W.(f\ W\ A) \subseteq A)$$

as an axiom.

Soundness and Completeness

$$\models^{QCL(ID)} \varphi \quad \text{iff} \quad ID \models^{\text{HOL}} \text{vld}\, \varphi_\tau$$

How meaningful is this cut-elimination result?

ID	Axiom Condition	$A \Rightarrow A$ $f(w,[A]) \subseteq [A]$
MP	Axiom Condition	$(A \Rightarrow B) \rightarrow (A \rightarrow B)$ $w \in [A] \rightarrow w \in f(w,[A])$
CS	Axiom Condition	$(A \wedge B) \rightarrow (A \Rightarrow B)$ $w \in [A] \rightarrow f(w,[A]) \subseteq \{w\}$
CEM	Axiom Condition	$(A \Rightarrow B) \vee (A \Rightarrow \neg B)$ $\lvert f(w,[A]) \rvert \leq 1$
AC	Axiom Condition	$(A \Rightarrow B) \wedge (A \Rightarrow C) \rightarrow (A \wedge C \Rightarrow B)$ $f(w,[A]) \subseteq [B] \rightarrow f(w,[A \wedge B]) \subseteq f(w,[A])$
RT	Axiom Condition	$(A \wedge B \Rightarrow C) \rightarrow ((A \Rightarrow B) \rightarrow (A \Rightarrow C))$ $f(w,[A]) \subseteq [B] \rightarrow f(w,[A]) \subseteq f(w,[A \wedge B])$
CV	Axiom Condition	$(A \Rightarrow B) \wedge \neg(A \Rightarrow \neg C) \rightarrow (A \wedge C \Rightarrow B)$ $(f(w,[A]) \subseteq [B]$ and $f(w,[A]) \cap [C] \neq \emptyset) \rightarrow f(w,[A \wedge C]) \subseteq [B]$
CA	Axiom Condition	$(A \Rightarrow B) \wedge (C \Rightarrow B) \rightarrow (A \vee C \Rightarrow B)$ $f(w,[A \vee B]) \subseteq f(w,[A]) \cup f(w,[B])$

Homework: Study cut-simulation for these axioms!

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Delta, fM(\lambda x \neg C \vee C)N}{\Delta, \neg\neg fM(\lambda x \neg C \vee C)N}\,\mathcal{G}(\neg) \qquad \dfrac{\dfrac{\Delta, \mathbf{C}}{\Delta, \neg\neg \mathbf{C}}\,\mathcal{G}(\neg) \quad \Delta * \neg\mathbf{C}}{\Delta, \neg(\neg C \vee C)}\,\mathcal{G}(\vee_{-})}{\Delta, \neg(\neg fM(\lambda x \neg C \vee C)N \vee (\neg C \vee C))}\,\mathcal{G}(\vee_{-})}{\Delta, \neg\Pi\lambda Y(\neg fM(\lambda x \neg C \vee C)Y \vee (\neg C \vee C))}\,\mathcal{G}(\Pi_{-}^{\mathbf{N}})}{\Delta, \neg\Pi\lambda A\Pi\lambda Y \neg fMAY \vee AY}\,\mathcal{G}(\Pi_{-}^{\lambda x \neg C \vee C})}{\dfrac{\Delta, \neg\Pi\lambda X\Pi\lambda A\Pi\lambda Y \neg fXAY \vee AY}{\Delta, \neg ID}\,\textit{Syn. Condition}}\,\mathcal{G}(\Pi_{-}^{\mathbf{M}})
$$

remove?